

PSEUDO-FUNCTORS MODELLING HIGHER STRUCTURES

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ABSTRACT. We introduce a new higher categorical structure called a weakly globular n -fold category. This structure is based on iterated internal categories and on the notion of weak globularity. We identify a suitable class of pseudo-functors whose strictification produces weakly globular n -fold categories.

1. INTRODUCTION

In this paper we introduce a new higher categorical structure called a weakly globular n -fold category. We show that this arises from pseudo-functors, a widely used notion in category theory and homotopy theory.

Weakly globular n -fold categories are a model of higher categories based on multi-simplicial sets. Simplicial and multi-simplicial models of higher categories have been developed over the years in several different contexts.

In the ground-breaking work of Lurie [22] and Joyal [18] on quasi-categories, simplicial sets with additional properties lead to $(\infty, 1)$ -categories; that is, higher categories in which the cells in dimension higher than 1 are invertible. Several Quillen equivalent model structures describe $(\infty, 1)$ -categories, including simplicial categories [8], Segal categories [8], complete Segal spaces [30], relative categories [3]. More recently, the notion of (∞, n) -categories emerged [29] [6], modelling higher categories where the cells in dimension higher than n are invertible. Several Quillen model structures on (∞, n) -categories have been developed, see [7]. Simplicial methods also feature prominently in describing weak ω -categories via complicial sets [36].

In this work we study higher categories in the truncated n -case. These structures generalize categories because in addition to objects and arrows they admit higher arrows (also called higher cells) and composition between them. When compositions are associative and unital, we obtain strict n -categories. The latter are insufficient for

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many applications of higher category theory. For instance, strict n -groupoids do not model n -types in dimension $n > 2$ (see [32] for a counterexample in dimension $n = 3$).

The wider class of weak n -categories is needed: in a weak n -category, compositions are associative and unital only up to an invertible cell in the next dimension and these associativity and unit isomorphisms are suitably compatible or coherent.

In dimension $n = 2$ and $n = 3$ the idea of weak n -category is embodied in the classical notions of bicategory [5] and tricategory [15]. In these structures, explicit diagrams encode the coherence axioms for the associativity and unit isomorphisms. Capturing the coherence axioms explicitly in dimension $n > 3$ seems untractable. Instead, different combinatorial machineries to define weak n -category [21] emerged: in these models the coherence data for the higher associativities are not given explicitly but they are automatically encoded in the combinatorics defining the models.

Different types of combinatorics have been used, including multi-simplicial sets as in Tamsamani and Simpson [32], [34], (higher) operads as in Batanin [4], Leinster [21] and Trimble [12], opetopes as in [1], [13] and several others.

In these classical models of higher categories the cells in dimension 0 up to n form a discrete structure, that is a set. We call this the *globularity condition* since it determines to the globular shape of the higher cells in the structure.

Weakly globular n -fold categories are based on a new paradigm to weaken higher categorical structure: the idea of *weak globularity*. In our approach the cells in dimension 0 up to n no longer form a set but have a higher categorical structure suitably equivalent to a discrete one. More precisely, they form a 'homotopically discrete n -fold category' as defined by the author in [25]. We call this the *weak globularity condition*.

In subsequent papers [27] [26] we show that weakly globular n -fold categories are suitably equivalent to the Tamsamani-Simpson model of weak n -categories [31] [34].

Weakly globular n -fold categories form a full subcategory of n -fold categories. These are, inductively, internal categories in $(n - 1)$ -fold categories. An n -fold category is therefore a 'rigid' structure in which all compositions are associative and unital.

The weakness in a weakly globular n -fold category is encoded by the weak globularity condition. The discretizations of the homotopically discrete structures in a weakly globular n -fold category play the role of sets of cells in the respective dimensions. Additional conditions are imposed in the definition of weakly globular n -fold category to obtain well behaved compositions of higher cells.

In the case $n = 2$, weakly globular double categories were introduced in joint work by the author in [9] and shown to be biequivalent to bicategories. The generalization to the case $n > 2$ is much more complex.

In previous work the author developed the notion of weakly globular n -fold structure in all dimension n in a homotopical context: for the modeling of connected n -types with weakly globular cat^n -groups [25], and for the modeling of general n -types with weakly globular n -fold groupoids [9].

This paper stretches far beyond a categorical generalization of the higher groupoidal case. In this paper we connect our new structure to pseudo-functors from a small category into the 2-category \mathbf{Cat} . In subsequent work [27], [26] this will lead to the proof of a suitable equivalence between weakly globular n -fold categories and the Tamsamani-Simpson model.

Pseudo-functors feature prominently in homotopy theory, for instance in iterated loop space theory [35]. They are also ubiquitous in category theory [10], and can be described with the language of 2-monad and their pseudo-algebras [28].

Weakly globular n -fold categories are a full subcategory of $(n-1)$ -fold simplicial objects in \mathbf{Cat} , that is functors $[\Delta^{n-1^{op}}, \mathbf{Cat}]$. We consider the pseudo-version of these, that is pseudo-functors $\mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}]$.

Crucial to this work is the use of the strictification of pseudo-functors into strict functors. This topic had many contributions in category theory, including [33]. We use in this work the elegant formulation of Power [28], further refined by Lack in [19]. The latter were recently generalized in [17], but for this work [28] and [19] are sufficient.

The classical theory of strictification of pseudo-algebras [28], [19] affords the strictification functor

$$St : \mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}] \rightarrow [\Delta^{n-1^{op}}, \mathbf{Cat}] \quad (1)$$

left adjoint to the inclusion.

The coherence axioms in a pseudo-functor are reminiscent of the coherence data for the compositions of higher cells in a weak higher category. So it is natural to ask if a subcategory of pseudo-functors can model, in a suitable sense, higher structures. In this paper we positively answer this question by introducing a subcategory

$$\mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}] \subset \mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}]$$

of *Segalic pseudo-functors*. Our main result, Theorem 4.5 is that the classical strictification functor (1) restricts to a functor

$$L_n : \mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}] \rightarrow \mathbf{Cat}_{\mathbf{wg}}^n .$$

In a subsequent paper [27] we associate to a Tamsamani weak n -category a Segalic pseudo-functor and thus build a 'rigidification' functor from Tamsamani weak n -categories to weakly globular n -fold categories.

This paper is organized as follows. Section 2 contains some preliminaries on (multi) simplicial techniques as well as on pseudo-functors and their strictification.

Section 3 introduces weakly globular n -fold categories and n -equivalences between them, and discusses the main properties of this structure. In Proposition 3.16 we prove a criterion for a n -fold category to be weakly globular. This is crucial to prove the main result in the next section.

Section 4 introduces Segalic pseudo-functors. We show in Proposition 4.3 that an n -fold category levelwise equivalent to a Segalic pseudo-functor is weakly globular. This leads to the main result Theorem 4.5.

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2. PRELIMINARIES

In this section we review some basic simplicial techniques that we will use throughout the paper as well as some categorical background on pseudo-functors and their strictification. The material in this section is well-known, see for instance [10], [14], [28], [19].

2.1. Simplicial objects. Let Δ be the simplicial category and let $\Delta^{n^{op}}$ denote the product of n copies of Δ^{op} . Given a category \mathcal{C} , $[\Delta^{n^{op}}, \mathcal{C}]$ is called the category of n -simplicial objects in \mathcal{C} (simplicial objects in \mathcal{C} when $n = 1$).

Notation 2.1. If $X \in [\Delta^{n^{op}}, \mathcal{C}]$ and $\underline{k} = ([k_1], \dots, [k_n]) \in \Delta^{n^{op}}$, we shall denote $X([k_1], \dots, [k_n])$ by $X(k_1, \dots, k_n)$, as well as X_{k_1, \dots, k_n} and $X_{\underline{k}}$. We shall also denote $\underline{k}(1, i) = ([k_1], \dots, [k_{i-1}], 1, [k_{i+1}], \dots, [k_n]) \in \Delta^{n^{op}}$ for $1 \leq i \leq n$.

Every n -simplicial object in \mathcal{C} can be regarded as a simplicial object in $[\Delta^{n-1^{op}}, \mathcal{C}]$ in n possible ways. For each $1 \leq i \leq n$ there is an isomorphism

$$\xi_i : [\Delta^{n^{op}}, \mathcal{C}] \rightarrow [\Delta^{op}, [\Delta^{n-1^{op}}, \mathcal{C}]]$$

given by

$$(\xi_i X)_r(k_1, \dots, k_{n-1}) = X(k_1, \dots, k_{i-1}, r, k_{i+1}, \dots, k_{n-1})$$

for $X \in [\Delta^{n^{op}}, \mathcal{C}]$ and $r \in \Delta^{op}$.

Definition 2.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, \mathcal{I} a small category. Denote

$$\overline{F} : [\mathcal{I}, \mathcal{C}] \rightarrow [\mathcal{I}, \mathcal{D}]$$

the functor given by

$$(\overline{F}X)_i = F(X(i))$$

for all $i \in \mathcal{I}$.

Definition 2.3. Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in any category \mathcal{C} with pullbacks. For each $1 \leq j \leq k$ and $k \geq 2$, let $\nu_j : X_k \rightarrow X_1$ be induced by the map $[1] \rightarrow [k]$ in Δ sending 0 to $j-1$ and 1 to j . Then the following diagram commutes:

$$\begin{array}{ccccccc}
 & & & & X_k & & \\
 & & \swarrow^{\nu_1} & \searrow^{\nu_2} & & \searrow^{\nu_k} & \\
 & X_1 & & X_1 & & \dots & X_1 \\
 \swarrow^{d_1} & & \searrow^{d_0} & \swarrow^{d_1} & \searrow^{d_0} & \dots & \swarrow^{d_1} & \searrow^{d_0} \\
 X_0 & & X_0 & & X_0 & \dots & X_0 & & X_0
 \end{array} \quad (2)$$

If $X_1 \times_{X_0} \cdots \times_{X_0}^k X_1$ denotes the limit of the lower part of the diagram (2), the k -th Segal map for X is the unique map

$$\mu_k : X_k \rightarrow X_1 \times_{X_0} \cdots \times_{X_0}^k X_1$$

such that $\text{pr}_j \mu_k = \nu_j$ where pr_j is the j^{th} projection.

Definition 2.4. Let $X \in [\Delta^{op}, \mathcal{C}]$ and suppose that there is a map $\gamma : X_0 \rightarrow X_0^d$ in \mathcal{C} such that the limit of the diagram

$$\begin{array}{ccccccc}
 & & & & X_1 & & X_1 & \dots & k & \dots & X_1 \\
 & & \swarrow^{\gamma d_1} & \searrow^{\gamma d_0} & \swarrow^{\gamma d_1} & \searrow^{\gamma d_0} & & & & \swarrow^{\gamma d_1} & \searrow^{\gamma d_0} \\
 & X_0^d & & X_0^d & & X_0^d & \dots & & & \dots & X_0^d & & X_0^d
 \end{array}$$

exists; denote the latter by $X_1 \times_{X_0^d} \cdots \times_{X_0^d}^k X_1$. Then the following diagram commutes, where ν_j is as in Definition 2.3, and $k \geq 2$

$$\begin{array}{ccccccc}
 & & & & X_k & & \\
 & & \swarrow^{\nu_1} & \searrow^{\nu_2} & & \searrow^{\nu_k} & \\
 & X_1 & & X_1 & & \dots & X_1 \\
 \swarrow^{\gamma d_1} & & \searrow^{\gamma d_0} & \swarrow^{\gamma d_1} & \searrow^{\gamma d_0} & \dots & \swarrow^{\gamma d_1} & \searrow^{\gamma d_0} \\
 X_0^d & & X_0^d & & X_0^d & \dots & X_0^d & & X_0^d
 \end{array}$$

The k -th induced Segal map for X is the unique map

$$\hat{\mu}_k : X_k \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

such that $\text{pr}_j \hat{\mu}_k = \nu_j$ where pr_j is the j^{th} projection.

2.2. n -Fold internal categories. Let \mathcal{C} be a category with finite limits. An internal category X in \mathcal{C} is a diagram in \mathcal{C}

$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s} \end{array} X_0 \quad (3)$$

where m, d_0, d_1, s satisfy the usual axioms of a category (see for instance [10]) for details. An internal functor is a morphism of diagrams like (3) commuting in the obvious way. We denote by $\text{Cat } \mathcal{C}$ the category of internal categories and internal functors.

The category $\text{Cat}^n(\mathcal{C})$ of n -fold categories in \mathcal{C} is defined inductively by iterating n times the internal category construction. That is, $\text{Cat}^1(\mathcal{C}) = \text{Cat}$ and, for $n > 1$,

$$\text{Cat}^n(\mathcal{C}) = \text{Cat}(\text{Cat}^{n-1}(\mathcal{C})).$$

When $\mathcal{C} = \text{Set}$, $\text{Cat}^n(\text{Set})$ is simply denoted by Cat^n and called the category of n -fold categories (double categories when $n = 2$).

2.3. Nerve functors. There is a nerve functor

$$N : \text{Cat } \mathcal{C} \rightarrow [\Delta^{op}, \mathcal{C}]$$

such that, for $X \in \text{Cat } \mathcal{C}$

$$(NX)_k = \begin{cases} X_0, & k = 0; \\ X_1, & k = 1; \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1, & k > 1. \end{cases}$$

When no ambiguity arises, we shall sometimes denote $(NX)_k$ by X_k for all $k \geq 0$.

The following fact is well known:

Proposition 2.5. *A simplicial object in \mathcal{C} is the nerve of an internal category in \mathcal{C} if and only if all the Segal maps are isomorphisms.*

By iterating the nerve construction, we obtain the multinerve functor

$$N_{(n)} : \text{Cat}^n(\mathcal{C}) \rightarrow [\Delta^{n^{op}}, \mathcal{C}].$$

Definition 2.6. An internal n -fold category $X \in \text{Cat}^n(\mathcal{C})$ is said to be discrete if $N_{(n)}X$ is a constant functor.

Each object of $\text{Cat}^n(\mathcal{C})$ can be considered as an internal category in $\text{Cat}^{n-1}(\mathcal{C})$ in n possible ways, corresponding to the n simplicial directions of its multinerve. To prove this, we use the following lemma, which is a straightforward consequence of the definitions.

Lemma 2.7.

- a) $X \in [\Delta^{n^{op}}, \mathcal{C}]$ is the multinerve of an n -fold category in \mathcal{C} if and only if, for all $1 \leq r \leq n$ and $[p_1], \dots, [p_r] \in \Delta^{op}$, $p_r \geq 2$

$$\begin{aligned} X(p_1, \dots, p_r, -) &\cong \\ &\cong X(p_1, \dots, p_{r-1}, 1, -) \times_{X(p_1, \dots, p_{r-1}, 0, -)} \cdots \times_{X(p_1, \dots, p_{r-1}, 0, -)} X(p_1, \dots, p_{r-1}, 1, -) \end{aligned} \quad (4)$$

- b) Let $X \in \mathbf{Cat}^n(\mathcal{C})$. For each $1 \leq k \leq n$, $[i] \in \Delta^{op}$, there is $X_i^{(k)} \in \mathbf{Cat}^{n-1}(\mathcal{C})$ with

$$N_{(n-1)} X_i^{(k)}(p_1, \dots, p_{n-1}) = N_{(n)} X(p_1, \dots, p_{k-1}, i, p_k, \dots, p_{n-1})$$

Proof.

a) By induction on n . By Proposition 2.5, it is true for $n = 1$. Suppose it holds for $n - 1$ and let $X \in \mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathcal{C}))$ with objects of objects (resp. arrows) X_0 (resp. X_1); denote $(NX)_p = X_p$. By definition of the multinerve

$$(N_{(n)} X)(p_1, \dots, p_r, -) = N_{(n-1)} X_{p_1}(p_2, \dots, p_r, -).$$

Hence using the induction hypothesis

$$\begin{aligned} N_{(n)} X(p_1 \dots p_r -) &= N_{(n-1)} X_{p_1}(p_2 \dots p_r -) \cong \\ &\cong N_{(n-1)} X_{p_1}(p_2 \dots p_{r-1} 1 -) \times_{N_{(n-1)} X_{p_1}(p_2 \dots p_{r-1} 0 -)} \cdots \times_{N_{(n-1)} X_{p_1}(p_2 \dots p_{r-1} 0 -)} N_{(n-1)} X_{p_1}(p_2 \dots p_{r-1} 1 -) = \\ &= N_{(n)} X(p_1 \dots p_{r-1} 1 -) \times_{N_{(n)} X(p_1 \dots p_{r-1} 0 -)} \cdots \times_{N_{(n)} X(p_1 \dots p_{r-1} 0 -)} N_{(n)} X(p_1 \dots p_{r-1} 1 -). \end{aligned}$$

Conversely, suppose $X \in [\Delta^{n^{op}}, \mathcal{C}]$ satisfies (4). Then for each $[p] \in \Delta^{op}$, $X(p, -)$ satisfies (4), hence

$$X(p, -) = N_{(n-1)} X_p$$

for $X_p \in \mathbf{Cat}^{n-1}(\mathcal{C})$. Also, by induction hypothesis

$$X(p, -) = X(1, -) \times_{X(0, -)} \cdots \times_{X(0, -)} X(1, -).$$

Thus we have the object $X \in \mathbf{Cat}^n(\mathcal{C})$ with objects X_0 , arrows X_1 and $X_p = X(p, -)$ as above.

- b) By part a), there is an isomorphism for $p_r \geq 2$

$$\begin{aligned} N_{(n)} X(p_1 \dots p_n) &= \\ &= N_{(n)} X(p_1 \dots p_{r-1} 1 \dots p_n) \times_{N_{(n)} X(p_1 \dots p_{r-1} 0 \dots p_n)} \cdots \times_{N_{(n)} X(p_1 \dots p_{r-1} 0 \dots p_n)} N_{(n)} X(p_1 \dots p_{r-1} 1 \dots p_n). \end{aligned}$$

In particular, evaluating this at $p_k = i$, this is saying the $(n - 1)$ -simplicial group taking $(p_1 \dots p_n)$ to $N_{(n)} X(p_1 \dots p_{k-1} i \dots p_{n-1})$ satisfies condition (4) in part a). Hence by part a) there exists $X_i^{(k)}$ with

$$N_{(n-1)} X_i^{(k)}(p_1 \dots p_{n-1}) = N_{(n)} X(p_1 \dots p_{k-1} i \dots p_{n-1})$$

as required. \square

Proposition 2.8. *For each $1 \leq k \leq n$ there is an isomorphism $\xi_k : \mathbf{Cat}^n(\mathcal{C}) \rightarrow \mathbf{Cat}^n(\mathcal{C})$ which associates to $X = \mathbf{Cat}^n(\mathcal{C})$ an object $\xi_k X$ of $\mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathcal{C}))$ with*

$$(\xi_k X)_i = X_i^{(k)} \quad i = 0, 1$$

with $X_i^{(k)}$ as in Lemma 2.7.

Proof. Consider the object of $[\Delta^{op}, [\Delta^{n-1op}, \mathcal{C}]]$ taking i to the $(n-1)$ -simplicial object associating to (p_1, \dots, p_{n-1}) the object

$$N_{(n)}X(p_1 \dots p_{k-1} \ i \ p_{k+1} \dots p_{n-1}) .$$

By Lemma 2.7 b), the latter is the multinerve of $X_i^{(k)} \in \mathbf{Cat}^{n-1}(\mathcal{C})$. Further, by Lemma 2.7 a), we have

$$N_{(n-1)}X_i^{(k)} \cong N_{(n-1)}X_1^{(k)} \times_{N_{(n-1)}X_0^{(k)}} \cdots \times_{N_{(n-1)}X_0^{(k)}} N_{(n-1)}X_1^{(k)} .$$

Hence $N_{(n)}X$ as a simplicial object in $[\Delta^{n-1op}, \mathcal{C}]$ along the k^{th} direction, has

$$(N_{(n)}X)_i = \begin{cases} N_{(n-1)}X_i^{(k)}, & i = 0, 1; \\ N_{(n-1)}(X_1^{(k)} \times_{X_0^{(k)}} \cdots \times_{X_0^{(k)}} X_1^{(k)}), & \text{for } i \geq 2. \end{cases}$$

This defines $\xi_k X \in \mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathcal{C}))$ with

$$(\xi_k X)_i = N_{(n-1)}X_i^{(k)} \quad i = 0, 1 .$$

We now define the inverse for ξ_k . Let $X \in \mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathcal{C}))$, and let $X_i = X_1 \times_{X_0} \cdots \times_{X_0} X_1$ for $i \geq 2$. The n -simplicial object X_k taking (p_1, \dots, p_n) to

$$N_{(n)}X_{p_k}(p_1 \dots p_{k-1} p_{k+1} \dots p_n)$$

satisfies condition (4), as easily seen. Hence by Lemma 2.7 there is $\xi'_k X \in \mathbf{Cat}^n \mathcal{C}$ such that $N_{(n)}\xi'_k X = X_k$. It is immediate to check that ξ_k and ξ'_k are inverse bijections. \square

Definition 2.9. The nerve functor in the k^{th} direction is defined as the composite

$$N^{(k)} : \mathbf{Cat}^n(\mathcal{C}) \xrightarrow{\xi_k} \mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathcal{C})) \xrightarrow{N} [\Delta^{op}, \mathbf{Cat}^{n-1}(\mathcal{C})]$$

so that, in the above notation,

$$(N^{(k)}X)_i = X_i^{(k)} \quad i = 0, 1 .$$

Note that $N_{(n)} = N^{(n)} \dots N^{(2)} N^{(1)}$.

Notation 2.10. When $\mathcal{C} = \mathbf{Set}$ we shall denote

$$J_n = N^{(n-1)} \dots N^{(1)} : \mathbf{Cat}^n \rightarrow [\Delta^{n-1op}, \mathbf{Cat}] .$$

Thus J_n amounts to taking the nerve construction in all but the last simplicial direction. Thus J_n amounts to taking the nerve construction in all but the last simplicial direction. The functor J_n is fully faithful, thus we can identify \mathbf{Cat}^n with the image $J_n(\mathbf{Cat}^n)$ of the functor J_n .

Given $X \in \mathbf{Cat}^n$, when no ambiguity arises we shall denote, for each $(s_1, \dots, s_{n-1}) \in \Delta^{n-1^{op}}$

$$X_{s_1, \dots, s_{n-1}} = (J_n X)_{s_1, \dots, s_{n-1}} \in \mathbf{Cat}$$

and more generally, if $1 \leq j \leq n-1$,

$$X_{s_1, \dots, s_j} = (N^{(j)} \dots N^{(1)} X)_{s_1, \dots, s_j} \in \mathbf{Cat}^{n-j}.$$

Let $ob : \mathbf{Cat} \mathcal{C} \rightarrow \mathcal{C}$ be the object of object functor. The left adjoint to ob is the discrete internal category functor d . By Proposition 2.8 we then have

$$\mathbf{Cat}^n \mathcal{C} \xrightarrow[\xi_n]{d} \mathbf{Cat}(\mathbf{Cat}^{n-1} \mathcal{C}) \xrightleftharpoons[d]{ob} \mathbf{Cat}^{n-1} \mathcal{C}.$$

We denote

$$d^{(n)} = \xi_n^{-1} \circ d \text{ for } n > 1, \quad d^{(1)} = d.$$

Thus $d^{(n)}$ is the discrete inclusion of $\mathbf{Cat}^{n-1} \mathcal{C}$ into $\mathbf{Cat}^n \mathcal{C}$ in the n^{th} direction.

The following is a characterization of objects of $[\Delta^{n-1^{op}}, \mathbf{Cat}]$ in the image of the functor J_n in 2.10.

Lemma 2.11. *Let $L \in [\Delta^{n-1^{op}}, \mathbf{Cat}]$ be such that, for all $\underline{k} \in \Delta^{n-1^{op}}$, $1 \leq i \leq n-1$ and $k_i \geq 2$, the Segal maps are isomorphisms:*

$$L_{\underline{k}} \cong L_{\underline{k}(1,i)} \times_{L_{\underline{k}(0,i)}} \cdots \times_{L_{\underline{k}(0,i)}} L_{\underline{k}(1,i)}. \quad (5)$$

Then $L \in \mathbf{Cat}^n$.

Proof. By induction on n . When $n = 2$, $L \in [\Delta^{op}, \mathbf{Cat}]$, $k \in \Delta^{op}$, $i = 1$, $\underline{k}(1, i) = 1$, $\underline{k}(0, i) = 0$, $k_i = k = 2$ and

$$L_k \cong L_1 \times_{L_0} \cdots \times_{L_0} L_1.$$

Thus by Proposition 2.5, $L \in \mathbf{Cat}^2$.

Suppose the lemma holds for $(n-1)$ and let $L \in [\Delta^{n-1^{op}}, \mathbf{Cat}]$ be as in the hypothesis. Consider $L_j \in [\Delta^{n-2^{op}}, \mathbf{Cat}]$ for $j \geq 0$. Let $\underline{r} \in \Delta^{n-2^{op}}$ and denote $\underline{k} = (j, \underline{r}) \in \Delta^{n-1^{op}}$. Then, for any $2 \leq i \leq n-1$, $k_i = r_{i-1}$ and

$$L_{\underline{k}} = (L_j)_{\underline{r}}, \quad L_{\underline{k}(1,i)} = (L_j)_{\underline{r}(1,i-1)}, \quad L_{\underline{k}(0,i)} = (L_j)_{\underline{r}(0,i-1)}.$$

Therefore (5) implies

$$(L_j)_{\underline{r}} = (L_j)_{\underline{r}(1,i-1)} \times_{(L_j)_{\underline{r}(0,i-1)}} \cdots \times_{(L_j)_{\underline{r}(0,i-1)}} (L_j)_{\underline{r}(1,i-1)}.$$

This means that L_j satisfies the inductive hypothesis and therefore $L_j \in \mathbf{Cat}^{n-1}$.

Taking $i = 1$ in (5) we see that, for each $k_1 \geq 2$ and $\underline{r} = (k_2, \dots, k_{n-1} \in \Delta^{n-2^{op}}$,

$$(L_{k_1})_{\underline{r}} = (L_{k_1})_{\underline{r}} = (L_1)_{\underline{r} \times (L_0)_{\underline{r}}} \cdot \dots \cdot \times_{(L_0)_{\underline{r}}}^{k_1} (L_1)_{\underline{r}}.$$

That is, we have isomorphisms in \mathbf{Cat}^{n-1}

$$L_{k_1} \cong L_1 \times_{L_0} \cdot \dots \cdot \times_{L_0}^{k_1} L_1.$$

We conclude from Proposition 2.5 the $L \in \mathbf{Cat}^n$.

□

2.4. Some functors on \mathbf{Cat} . The connected component functor

$$q : \mathbf{Cat} \rightarrow \mathbf{Set}$$

associates to a category its set of paths components. This is left adjoint to the discrete category functor

$$d^{(1)} : \mathbf{Set} \rightarrow \mathbf{Cat}$$

associating to a set X the discrete category on that set. We denote by

$$\gamma^{(1)} : \text{Id} \Rightarrow d^{(1)}q$$

the unit of the adjunction $q \dashv d^{(1)}$.

Lemma 2.12. *q preserves fiber products over discrete objects and sends equivalences of categories to isomorphisms.*

Proof. We claim that q preserves products; that is, given categories \mathcal{C} and \mathcal{D} , there is a bijection

$$q(\mathcal{C} \times \mathcal{D}) = q(\mathcal{C}) \times q(\mathcal{D}).$$

In fact, given $(c, d) \in q(\mathcal{C} \times \mathcal{D})$ the map $q(\mathcal{C} \times \mathcal{D}) \rightarrow q(\mathcal{C}) \times q(\mathcal{D})$ given by $[(c, d)] = ([c], [d])$ is well defined and is clearly surjective. On the other hand, this map is also injective: given $[(c, d)]$ and $[(c', d')]$ with $[c] = [c']$ and $[d] = [d']$, we have paths in \mathcal{C}

$$\begin{array}{ccccc} c & \text{---} & \dots & \text{---} & c' \\ d & \text{---} & \dots & \text{---} & d' \end{array}$$

and hence a path in $\mathcal{C} \times \mathcal{D}$

$$(c, d) \text{ --- } \dots \text{ --- } (c', d) \text{ --- } \dots \text{ --- } (c', d').$$

Thus $[(c, d)] = [(c', d')]$ and so the map is also injective, hence it is a bijection, as claimed.

Given a diagram in \mathbf{Cat} $\mathcal{C} \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{D}$ with \mathcal{E} discrete, we have

$$\mathcal{C} \times_{\mathcal{E}} \mathcal{D} = \coprod_{x \in \mathcal{E}} \mathcal{C}_x \times \mathcal{D}_x \quad (6)$$

where \mathcal{C}_x , \mathcal{D}_x are the full subcategories of \mathcal{C} and \mathcal{D} with objects c , d such that $f(c) = x = g(d)$. Since q preserves products and (being left adjoint) coproducts, we conclude by (6) that

$$q(\mathcal{C} \times_{\mathcal{E}} \mathcal{D}) \cong q(\mathcal{C}) \times_{\mathcal{E}} q(\mathcal{D}) .$$

Finally, if $F : \mathcal{C} \simeq \mathcal{D} : G$ is an equivalence of categories, $FG\mathcal{C} \cong \mathcal{C}$ and $FG\mathcal{D} \cong \mathcal{D}$ which implies that $qFqG\mathcal{C} \cong q\mathcal{C}$ and $qFqG\mathcal{D} \cong q\mathcal{D}$, so $q\mathcal{C}$ and $q\mathcal{D}$ are isomorphic. \square

The isomorphism classes of objects functors

$$p : \mathbf{Cat} \rightarrow \mathbf{Set}$$

associates to a category the set of isomorphism classes of its objects. Notice that if \mathcal{C} is a groupoid, $p\mathcal{C} = q\mathcal{C}$.

Lemma 2.13. *p preserves pullbacks over discrete objects and sends equivalences of categories to isomorphisms.*

Proof. For a category \mathcal{C} , let $m\mathcal{C}$ be its maximal subgroupoid. Then $p\mathcal{C} = qm\mathcal{C}$. Given a diagram in \mathbf{Cat} $\mathcal{C} \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{D}$ with \mathcal{E} discrete, we have

$$\mathcal{C} \times_{\mathcal{E}} \mathcal{D} = \coprod_{x \in \mathcal{E}} \mathcal{C}_x \times \mathcal{D}_x .$$

Since, as easily seen, m commutes with (co)products, and $m\mathcal{E} = \mathcal{E}$, we obtain $m(\mathcal{C} \times_{\mathcal{E}} \mathcal{D}) = m\mathcal{C} \times_{\mathcal{E}} m\mathcal{D}$; so by Lemma 2.12,

$$p(\mathcal{C} \times_{\mathcal{E}} \mathcal{D}) = qm(\mathcal{C} \times_{\mathcal{E}} \mathcal{D}) = q(m\mathcal{C} \times_{\mathcal{E}} m\mathcal{D}) = qm\mathcal{C} \times_{q\mathcal{E}} qm\mathcal{D} = p\mathcal{C} \times_{\mathcal{E}} p\mathcal{D} .$$

Finally, if $F : \mathcal{C} \simeq \mathcal{D} : G$ is an equivalence of categories, $FG\mathcal{C} \cong \mathcal{C}$ and $FG\mathcal{D} \cong \mathcal{D}$ which implies that $pFpG\mathcal{C} \cong p\mathcal{C}$ and $qFqG\mathcal{D} \cong q\mathcal{D}$, so $q\mathcal{C}$ and $q\mathcal{D}$ are isomorphic. \square

2.5. Pseudo-functors and their strictification. The functor 2-category $[\Delta^{n^{op}}, \mathbf{Cat}]$ is 2-monadic over $[ob(\Delta^{n^{op}}), \mathbf{Cat}]$ where $ob(\Delta^{n^{op}})$ is the set of objects of $\Delta^{n^{op}}$. Let

$$U : [\Delta^{n^{op}}, \mathbf{Cat}] \rightarrow [ob(\Delta^{n^{op}}), \mathbf{Cat}]$$

be the forgetful functor $(UX)_{\underline{k}} = X_{\underline{k}}$. Its left adjoint F is given on objects by

$$(FH)_{\underline{k}} = \coprod_{\underline{r} \in ob(\Delta^{n^{op}})} \Delta^{n^{op}}(\underline{r}, \underline{k}) \times H_{\underline{r}}$$

for $H \in [ob(\Delta^{n-1^{op}}), \mathbf{Cat}]$, $\underline{k} \in ob(\Delta^{n-1^{op}})$. If T is the monad corresponding to the adjunction $F \dashv U$, then

$$(TH)_{\underline{k}} = \coprod_{\underline{r} \in ob(\Delta^{n^{op}})} \Delta^{n^{op}}(\underline{r}, \underline{k}) \times H_{\underline{r}}$$

A pseudo T -algebra is given by $H \in [ob(\Delta^{n^{op}}), \mathbf{Cat}]$,

$$h_n : \coprod_{\underline{r} \in ob(\Delta^{n^{op}})} \Delta^{n^{op}}(\underline{r}, \underline{k}) \times H_{\underline{r}} \rightarrow H_{\underline{k}}$$

and additional data, as described in [28]. This amounts precisely to functors from $\Delta^{n^{op}}$ to \mathbf{Cat} and the 2-category $\mathbf{Ps}\text{-}T\text{-alg}$ of pseudo T -algebras corresponds to the 2-category $\mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}]$ of pseudo-functors, pseudo-natural transformations and modifications.

The strictification result proved in [28] yields that every pseudo-functor from $\Delta^{n^{op}}$ to \mathbf{Cat} is equivalent, in $\mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}]$, to a 2-functor.

Given a pseudo T -algebra as above, [28] consider the factorization of $h : TH \rightarrow H$ as

$$TH \xrightarrow{v} L \xrightarrow{g} H$$

with $v_{\underline{k}}$ bijective on objects and $g_{\underline{k}}$ fully faithful, for each $\underline{k} \in \Delta^{n^{op}}$. It is shown in [28] that it is possible to give a strict T -algebra structure $TL \rightarrow L$ such that (g, Tg) is an equivalence of pseudo T -algebras. It is immediate to see that, for each $\underline{k} \in \Delta^{n^{op}}$, $g_{\underline{k}}$ is an equivalence of categories.

Further, it is shown in [19] that $St : \mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}] \rightarrow [\Delta^{n^{op}}, \mathbf{Cat}]$ as described above is left adjoint to the inclusion

$$J : [\Delta^{n^{op}}, \mathbf{Cat}] \rightarrow \mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}]$$

and that the components of the units are equivalences in $\mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}]$.

3. WEAKLY GLOBULAR N -FOLD CATEGORIES

In this section we define weakly globular n -fold categories and establish their main properties.

The definition of weakly globular n -fold category uses the notion of homotopically discrete n -fold category from [24] in order to formulate the weak globularity condition. We recall this notion and its main properties in Section 3.1.

In Section 3.2 we inductively define of weakly globular n -fold categories and of n -equivalences between them. In Section 3.3 we establish the main properties of weakly globular n -fold categories. We show in Proposition 3.16 b) a criterion for a n -fold category to be weakly globular playing a crucial role in the proof of the main result Theorem 4.5.

3.1. Homotopically discrete n -fold categories.

Definition 3.1. Define inductively the full subcategory $\mathbf{Cat}_{\text{hd}}^n \subset \mathbf{Cat}^n$ of homotopically discrete n -fold categories.

For $n = 1$, $\mathbf{Cat}_{\text{hd}}^1 = \mathbf{Cat}_{\text{hd}}$ is the category of equivalence relations. Denote by $p^{(1)} = p : \mathbf{Cat} \rightarrow \mathbf{Set}$ the isomorphism classes of object functor.

Suppose, inductively, that for each $1 \leq k \leq n-1$ we defined $\mathbf{Cat}_{\text{hd}}^k \subset \mathbf{Cat}^k$ and k -equivalences such that the following holds:

- a) The k^{th} direction in $\mathbf{Cat}_{\text{hd}}^k$ is groupoidal; that is, if $X \in \mathbf{Cat}_{\text{hd}}^k$, $\xi_k X \in \mathbf{Gpd}(\mathbf{Cat}^{k-1})$ (where $\xi_k X$ is as in Proposition 2.8).
- b) There is a functor $p^{(k)} : \mathbf{Cat}_{\text{hd}}^k \rightarrow \mathbf{Cat}_{\text{hd}}^{k-1}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbf{Cat}_{\text{hd}}^k & \xrightarrow{N^{(k-1)} \dots N^{(1)}} & [\Delta^{k-1 \text{op}}, \mathbf{Cat}] \\ p^{(k)} \downarrow & & \downarrow \bar{p} \\ \mathbf{Cat}_{\text{hd}}^{k-1} & \xrightarrow{N_{(k-1)}} & [\Delta^{k-1 \text{op}}, \mathbf{Set}] \end{array} \quad (7)$$

Note that this implies that $(p^{(k)} X)_{s_1 \dots s_{k-1}} = p X_{s_1 \dots s_{k-1}}$ for all $(s_1 \dots s_{k-1}) \in \Delta^{k-1 \text{op}}$.

$\mathbf{Cat}_{\text{hd}}^n$ is the full subcategory of $[\Delta^{\text{op}}, \mathbf{Cat}_{\text{hd}}^{n-1}]$ whose objects X are such that

$$(i) \quad X_s \cong X_1 \times_{X_0} \dots \times_{X_0}^s X_1 \quad \text{for all } s \geq 2.$$

In particular this implies that $X \in \mathbf{Cat}(\mathbf{Gpd}(\mathbf{Cat}^{n-2})) = \mathbf{Gpd}(\mathbf{Cat}^{n-1})$ and the n^{th} direction in X is groupoidal.

- (ii) The functor

$$\bar{p}^{(n-1)} : \mathbf{Cat}_{\text{hd}}^n \subset [\Delta^{\text{op}}, \mathbf{Cat}_{\text{hd}}^{n-1}] \rightarrow [\Delta^{\text{op}}, \mathbf{Cat}_{\text{hd}}^{n-2}]$$

restricts to a functor

$$p^{(n)} : \mathbf{Cat}_{\text{hd}}^n \rightarrow \mathbf{Cat}_{\text{hd}}^{n-1}$$

Note that this implies that $(p^{(n)} X)_{s_1 \dots s_{n-1}} = p X_{s_1 \dots s_{n-1}}$ and that the following diagram commutes

$$\begin{array}{ccc} \mathbf{Cat}_{\text{hd}}^n & \xrightarrow{N^{(n-1)} \dots N^{(1)}} & [\Delta^{n-1 \text{op}}, \mathbf{Cat}] \\ p^{(n)} \downarrow & & \downarrow \bar{p} \\ \mathbf{Cat}_{\text{hd}}^{n-1} & \xrightarrow{N_{(n-1)}} & [\Delta^{n-1 \text{op}}, \mathbf{Set}] \end{array} \quad (8)$$

Definition 3.2. Denote by $\gamma_X^{(n)} : X \rightarrow d^{(n)} p^{(n)} X$ the morphism given by

$$(\gamma_X^{(n)})_{s_1 \dots s_{n-1}} : X_{s_1 \dots s_{n-1}} \rightarrow d p X_{s_1 \dots s_{n-1}}$$

for all $(s_1, \dots, s_{n-1}) \in \Delta^{n-1 \text{op}}$. Denote by

$$X^d = d^{(n)} d^{(n-1)} \dots d^{(1)} p^{(1)} p^{(2)} \dots p^{(n)} X$$

and by $\gamma_{(n)}$ the composite

$$X \xrightarrow{\gamma^{(n)}} d^{(n)} p^{(n)} X \xrightarrow{d^{(n)} \gamma^{(n-1)}} d^{(n)} d^{(n-1)} p^{(n-1)} p^{(n)} X \rightarrow \dots \rightarrow X^d.$$

For each $a, b \in X_0^d$ denote by $X(a, b)$ the fiber at (a, b) of the map

$$X_1 \xrightarrow{(d_0, d_1)} X_0 \times X_0 \xrightarrow{\gamma_{(n)} \times \gamma_{(n)}} X_0^d \times X_0^d .$$

Definition 3.3. Define inductively n -equivalences in $\mathbf{Cat}_{\text{hd}}^n$. For $n = 1$, a 1-equivalence is an equivalence of categories. Suppose we defined $(n - 1)$ -equivalences in $\mathbf{Cat}_{\text{hd}}^{n-1}$. Then a map $f : X \rightarrow Y$ in $\mathbf{Cat}_{\text{hd}}^n$ is an n -equivalence if, for all $a, b \in X_0^d$, $f(a, b) : X(a, b) \rightarrow Y(fa, fb)$ and $p^{(n)}f$ are $(n - 1)$ -equivalences.

The main properties of the category $\mathbf{Cat}_{\text{hd}}^n$ are summarized in the proposition below, whose proof can be found in [24]

Proposition 3.4. [24]

- a) $\gamma^{(n)}$ and $\gamma_{(n)}$ are $(n - 1)$ -equivalences.
- b) For each $X \in \mathbf{Cat}_{\text{hd}}^n$ the induced Segal maps

$$X_k \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

for each $k \geq 2$ are $(n - 1)$ -equivalences.

- c) $f : X \rightarrow Y$ in $\mathbf{Cat}_{\text{hd}}^n$ is an n -equivalence if and only if $X^d \cong Y^d$.
- d) The functor $J_n : \mathbf{Cat}^n \rightarrow [\Delta^{n-1^{op}}, \mathbf{Cat}]$ restricts to a functor $J_n : \mathbf{Cat}_{\text{hd}}^n \rightarrow [\Delta^{n-1^{op}}, \mathbf{Cat}_{\text{hd}}]$.

3.2. The definition of weakly globular n -fold categories. In this section we define the category $\mathbf{Cat}_{\text{wg}}^n$ of weakly globular n -fold categories and n -equivalences.

The idea of the definition is to build the structure by induction on dimension starting with the category \mathbf{Cat} with equivalences of categories.

At dimension n , the structure is a full subcategory of simplicial objects in $\mathbf{Cat}_{\text{wg}}^{n-1}$. Unraveling this definition, this affords an embedding

$$\mathbf{Cat}_{\text{wg}}^n \xhookrightarrow{J_n} [\Delta^{n-1^{op}}, \mathbf{Cat}] .$$

The first condition for $X \in [\Delta^{op}, \mathbf{Cat}_{\text{wg}}^{n-1}]$ to be an object of $\mathbf{Cat}_{\text{wg}}^n$ is the weak globularity condition that X_0 is homotopically discrete.

The set underlying the discrete $(n - 1)$ -fold category X_0^d plays the role of set of cells in dimension $(n - 1)$. When $0 \leq r \leq n - 1$, the set underlying $(J_n X)_{1 \dots r}^d$ corresponds to the set of r -cells.

The next condition in the definition of $\mathbf{Cat}_{\text{wg}}^n$ is that the Segal maps

$$X_k \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are isomorphisms for all $k \geq 2$. Since each $X_k \in \mathbf{Cat}^{n-1}$, by the characterization of internal categories via the Segal condition (Proposition 2.5) it follows that X is an n -fold category.

We further require the induced Segal map condition stating that, for each $k \geq 2$, the maps in $\mathbf{Cat}_{\mathbf{wg}}^{n-1}$

$$X_k \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

are $(n-1)$ -equivalences. This condition controls the compositions of higher cells and is the analogue of the Segal condition in the Tamsamani-Simpson model [34], [31].

We finally require the existence of a truncation functor $p^{(n)}$ from $\mathbf{Cat}_{\mathbf{wg}}^n$ to $\mathbf{Cat}_{\mathbf{wg}}^{n-1}$ obtained by applying dimensionwise the isomorphism classes of object functor to the corresponding diagram in $[\Delta^{n-1^{op}}, \mathbf{Cat}]$. In the case $n = 2$, this last condition is redundant.

The functor $p^{(n)}$ is used to define n -equivalences, thus completing the inductive step in the definition of $\mathbf{Cat}_{\mathbf{wg}}^n$. The definition of n -equivalences is given in terms of two conditions: the first is a higher dimensional generalization of the notion of fully faithfulness of a functor, the second is a generalization of 'essentially surjective on objects'.

Definition 3.5. For $n = 1$, $\mathbf{Cat}_{\mathbf{wg}}^1 = \mathbf{Cat}$ and 1-equivalences are equivalences of categories.

Suppose, inductively, that we defined $\mathbf{Cat}_{\mathbf{wg}}^{n-1}$ and $(n-1)$ -equivalences. Then $\mathbf{Cat}_{\mathbf{wg}}^n$ is the full subcategory of $[\Delta^{op}, \mathbf{Cat}_{\mathbf{wg}}^{n-1}]$ whose objects X are such that

- a) *Weak globularity condition* $X_0 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$.
- b) *Segal condition* For all $k \geq 2$ the Segal maps are isomorphisms:

$$X_k \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1 .$$

- c) *Induced Segal condition* For all $k \geq 2$ the induced Segal maps

$$X_k \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

(induced by the map $\gamma : X_0 \rightarrow X_0^d$) are $(n-1)$ -equivalences.

- d) *Truncation functor* There is a functor $p^{(n)} : \mathbf{Cat}_{\mathbf{hd}}^n \rightarrow \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ making the following diagram commute

$$\begin{array}{ccc} \mathbf{Cat}_{\mathbf{wg}}^n & \xrightarrow{J_n} & [\Delta^{n-1^{op}}, \mathbf{Cat}] \\ p^{(n)} \downarrow & & \downarrow \bar{p} \\ \mathbf{Cat}_{\mathbf{wg}}^{n-1} & \xrightarrow{N_{(n-1)}} & [\Delta^{n-1^{op}}, \mathbf{Set}] \end{array}$$

Given $a, b \in X_0^d$, denote by $X(a, b)$ the fiber at (a, b) of the map

$$X_1 \xrightarrow{(\partial_0, \partial_1)} X_0 \times X_0 \xrightarrow{\gamma \times \gamma} X_0^d \times X_0^d .$$

We say that a map $f : X \rightarrow Y$ in $\mathbf{Cat}_{\mathbf{wg}}^n$ is an n -equivalence if

i) For all $a, b \in X_0^d$

$$f(a, b) : X(a, b) \rightarrow Y(fa, fb)$$

is an $(n - 1)$ -equivalence.

ii) $p^{(n)}f$ is an $(n - 1)$ -equivalence.

This completes the inductive step in the definition of $\mathbf{Cat}_{\mathbf{wg}}^n$.

Remark 3.6. It follows by Definition 3.5, Definition 3.1 and Proposition 3.4 that $\mathbf{Cat}_{\mathbf{hd}}^n \subset \mathbf{Cat}_{\mathbf{wg}}^n$.

3.3. Properties of weakly globular n -fold categories. In this section we discuss the main properties of weakly globular n -fold categories. In Proposition 3.10 we show that a weakly globular n -fold category n -equivalent to a homotopically discrete one is homotopically discrete. This generalizes to higher dimension the fact that a category equivalent to a homotopically discrete one is homotopically discrete. We deduce in Corollary 3.11 a criterion for a weakly globular n -fold category to be homotopically discrete.

The main result of this section, Proposition 3.16 b), gives a criterion for an n -fold category to be weakly globular. This result will be used crucially in the proof of Proposition 4.3 to characterize n -fold categories levelwise equivalent to Segalic pseudo-functors. This leads to the main result Theorem 4.5 on the strictification of Segalic pseudo-functors.

The proof of Proposition 3.16 b) uses an inductive argument in conjunction with the proof of a property of the category $\mathbf{Cat}_{\mathbf{wg}}^n$ (Proposition 3.16 a)): the fact that the nerve functor in direction 2, when applied to $\mathbf{Cat}_{\mathbf{wg}}^n$ takes values in $[\Delta^{op}, \mathbf{Cat}_{\mathbf{wg}}^{n-1}]$.

Definition 3.7. For each $1 \leq j \leq n$ denote

$$\begin{aligned} p^{(j,n)} &= p^{(j)}p^{(j-1)} \dots p^{(n)} : \mathbf{Cat}_{\mathbf{wg}}^n \rightarrow \mathbf{Cat}_{\mathbf{wg}}^{j-1} \\ p^{(n,n)} &= p^{(n)}. \end{aligned}$$

Lemma 3.8. For each $X \in \mathbf{Cat}_{\mathbf{wg}}^n$, $1 \leq j < n$ and $s \geq 2$ it is

$$\begin{aligned} p^{(j,n-1)}X_s &\cong p^{(j,n-1)}(X_1 \times_{X_0} \dots \times_{X_0}^s X_1) = \\ &= p^{(j,n-1)}X_1 \times_{p^{(j,n-1)}X_0} \dots \times_{p^{(j,n-1)}X_0}^s p^{(j,n-1)}X_1. \end{aligned} \tag{9}$$

Proof. Since $X \in \mathbf{Cat}_{\mathbf{wg}}^n$ by definition $p^{(n)}X \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$, hence

$$p^{(n-1)}(X_1 \times_{X_0} \dots \times_{X_0}^s X_1) = p^{(n-1)}X_1 \times_{p^{(n-1)}X_0} \dots \times_{p^{(n-1)}X_0}^s p^{(n-1)}X_1$$

which is (9) for $j = n - 1$. Since $p^{(j+1,n)}X \in \mathbf{Cat}_{\mathbf{wg}}^j$ for $1 \leq j \leq (n - 1)$, its Segal maps are isomorphisms. Further for all $s \geq 0$

$$(p^{(j+1)} \dots p^{(n)}X)_s = p^{(j)} \dots p^{(n-1)}X_s = p^{(j,n-1)}X_s$$

with $X_s = X_1 \times_{X_0} \dots \times_{X_0}^s X_1$ for $s \geq 2$. This proves (9). \square

Remark 3.9. It follows immediately from Lemma 3.8 that if $X \in \mathbf{Cat}_{\mathbf{wg}}^n$, for all $s \geq 2$

$$X_{s0}^d = (X_{10} \times_{X_{00}} \cdots \times_{X_{00}} X_{10})^d = X_{10}^d \times_{X_{00}^d} \cdots \times_{X_{00}^d} X_{10}^d. \quad (10)$$

In fact, by (10) in the case $j = 2$, taking the 0-component, we obtain

$$\begin{aligned} p^{(1,n-2)}(X_{10} \times_{X_{00}} \cdots \times_{X_{00}} X_{10}) &= \\ &= p^{(1,n-2)} X_{10} \times_{p \cdots p^{(n-2)} X_{00}} \cdots \times_{p \cdots p^{(n-2)} X_{00}} p^{(1,n-2)} X_{10} \end{aligned}$$

which is the same as (10).

The following proposition is a higher dimensional generalization of the fact that, if a category is equivalent to an equivalence relation, it is itself an equivalence relation.

Proposition 3.10. *Let $f : X \rightarrow Y$ be a n -equivalence in $\mathbf{Cat}_{\mathbf{wg}}^n$ with $Y \in \mathbf{Cat}_{\mathbf{hd}}^n$, then $X \in \mathbf{Cat}_{\mathbf{hd}}^n$.*

Proof. By induction on n . It is clear for $n = 1$. Suppose it is true for $n - 1$ and let f be as in the hypothesis. Then $p^{(n)}f : p^{(n)}X \rightarrow p^{(n)}Y$ is a $(n - 1)$ -equivalence with $p^{(n)}Y \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ since $Y \in \mathbf{Cat}_{\mathbf{hd}}^n$. It follows by induction hypothesis that $p^{(n)}X \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$. We have

$$X_1 = \coprod_{a,b \in X_0^d} X(a, b). \quad (11)$$

Since f is a n -equivalence, there are $(n - 1)$ -equivalences

$$f(a, b) : X(a, b) \rightarrow Y(fa, fb)$$

where $Y(fa, fb) \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ since $Y \in \mathbf{Cat}_{\mathbf{hd}}^n$. By induction hypothesis, it follows that $X(a, b) \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$. From (11) and the fact that $\mathbf{Cat}_{\mathbf{hd}}^{n-1}$ is closed under coproducts (see [24], Lemma 3.8 a)), we conclude that $X_1 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$.

Since $X \in \mathbf{Cat}_{\mathbf{wg}}^n$, the induced Segal maps

$$\hat{\mu}_s : X_s = X_1 \times_{X_0} \cdots \times_{X_0} X_1 \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

is a $(n - 1)$ -equivalence. Since, from above, X_1 is homotopically discrete and X_0^d is discrete, by Lemma [24, Lemma 3.8 c],

$$X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$$

Thus by induction hypothesis applied to the induced Segal map $\hat{\mu}_s$ we conclude that $X_s \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ for all $s \geq 0$.

In summary, we showed that $X \in \mathbf{Cat}_{\mathbf{wg}}^n$ is such that $X_s \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ for all $s \geq 0$ and $p^{(n)}X \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$. Therefore, by definition, $X \in \mathbf{Cat}_{\mathbf{hd}}^n$. \square

Corollary 3.11. *Let $X \in \mathbf{Cat}_{\mathbf{wg}}^n$ be such that X_1 and $p^{(n)}X$ are in $\mathbf{Cat}_{\mathbf{hd}}^{n-1}$. Then $X \in \mathbf{Cat}_{\mathbf{hd}}^n$.*

Proof. Since $X \in \mathbf{Cat}_{\mathbf{wg}}^n$, the induced Segal maps

$$\hat{\mu}_s : X_s \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

are $(n-1)$ -equivalences for all $s \geq 2$. Since by hypothesis $X_1 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ and X_0^d is discrete, by (see [24], Lemma 3.8 a)),

$$X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}.$$

By Proposition 3.10 applied to $\hat{\mu}_s$ we conclude that $X_s \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ for all $s \geq 2$. Therefore $X \in \mathbf{Cat}_{\mathbf{wg}}^n$ is such that $X_s \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ and $p^{(n)}X \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$. By definition then $X \in \mathbf{Cat}_{\mathbf{hd}}^n$. \square

Corollary 3.12. *Let $X \in \mathbf{Cat}_{\mathbf{wg}}^n$, then $X \in \mathbf{Cat}_{\mathbf{hd}}^n$ if and only if there is an n -equivalence $\gamma : X \rightarrow Y$ with Y discrete.*

Proof. If $X \in \mathbf{Cat}_{\mathbf{hd}}^n$ then by Lemma [24, Lemma 3.8 a], $\gamma_{(n)} : X \rightarrow X^d$ is an n -equivalence. Conversely, suppose that there is an n -equivalence $\gamma : X \rightarrow Y$ with Y discrete, then in particular $Y \in \mathbf{Cat}_{\mathbf{hd}}^n$ so, by Proposition 3.10, $X \in \mathbf{Cat}_{\mathbf{hd}}^n$. \square

Definition 3.13. Given $X \in \mathbf{Cat}^n$ and $k \geq 0$, let $N^{(2)}X \in [\Delta^{op}, \mathbf{Cat}^{n-1}]$ as in Definition 2.9 so that for each $k \geq 0$, $(N^{(2)}X)_k = X_k^{(2)} \in [\Delta^{op}, \mathbf{Cat}^{n-2}]$ is given by

$$(X_k^{(2)})_s = \begin{cases} X_{0k}, & s = 0; \\ X_{1k}, & s = 1; \\ X_{sk} = X_{1k} \times_{X_{0k}} \cdots \times_{X_{0k}} X_{1k}, & s \geq 2. \end{cases}$$

We denote by $N_{(n)}\mathbf{Cat}_{\mathbf{wg}}^n$ the image of the multinerve functor $N_{(n)} : \mathbf{Cat}_{\mathbf{wg}}^n \rightarrow [\Delta^{n-1op}, \mathbf{Cat}]$. Note that, since $N_{(n)}$ is fully faithful, we have an isomorphism $\mathbf{Cat}_{\mathbf{wg}}^n \cong N_{(n)}\mathbf{Cat}_{\mathbf{wg}}^n$.

The following lemmas are needed in the initial steps in the induction in the proof of Proposition 3.16.

Lemma 3.14. *Let $X \in \mathbf{Cat}^2$ be such that*

- i) $X_0 \in \mathbf{Cat}_{\mathbf{hd}}$,
- ii) $\bar{p}J_2X \in N\mathbf{Cat}$.

Then $X \in \mathbf{Cat}_{\mathbf{wg}}^2$.

Proof. Since $X_0 \in \mathbf{Cat}_{\mathbf{hd}}$, $pX_0 = X_0^d$. By hypothesis, $pX_2 \cong pX_1 \times_{pX_0} pX_1$ and $X_2 \cong X_1 \times_{X_0} X_1$. Using the fact that p commutes with pullbacks over discrete objects, we obtain

$$p(X_1 \times_{X_0} X_1) \cong pX_2 \cong pX_1 \times_{pX_0} pX_1 = pX_1 \times_{pX_0^d} pX_1 = p(X_1 \times_{X_0^d} X_1).$$

This shows that the map

$$\hat{\mu}_2 : X_1 \times_{X_0} X_1 \rightarrow X_1 \times_{X_0^d} X_1$$

is essentially surjective on objects. On the other hand, this map is also fully faithful. In fact, given $(a, b), (c, d) \in X_{10} \times_{X_{00}} X_{10}$, we have

$$\begin{aligned} (X_1 \times_{X_0} X_1)((a, b), (c, d)) &\cong X_1(a, c) \times_{X_0(\partial_0 a, \partial_0 c)} X_1(b, d) \cong \\ &\cong X_1(a, c) \times X_1(b, d) \cong (X_1 \times_{X_0^d} X_1)(\hat{\mu}_2(a, b), \hat{\mu}_2(c, d)) \end{aligned}$$

where we used the fact that $X_0(\partial_0 a, \partial_0 c)$ is the one-element set, since $X_0 \in \mathbf{Cat}_{\text{hd}}$. We conclude that $\hat{\mu}_2$ is an equivalence of categories.

Similarly one shows that for all $k \geq 2$

$$\hat{\mu}_k : X_1 \times_{X_0} \cdots \times_{X_0} X_1 \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

is an equivalence of categories. By definition, this means that $X \in \mathbf{Cat}_{\text{wg}}^2$. \square

Lemma 3.15.

- a) The functor $N^{(2)} : \mathbf{Cat}^3 \rightarrow [\Delta^{op}, \mathbf{Cat}^2]$ restricts to $N^{(2)} : \mathbf{Cat}_{\text{wg}}^3 \rightarrow [\Delta^{op}, \mathbf{Cat}_{\text{wg}}^2]$.
- b) Let $X \in \mathbf{Cat}^3$ be such that
 - i) $X_0 \in \mathbf{Cat}_{\text{hd}}^2$, $X_{s0} \in \mathbf{Cat}_{\text{hd}}$.
 - ii) $\bar{p}J_3X \in N_{(2)}\mathbf{Cat}_{\text{wg}}^2$.

Then $X \in \mathbf{Cat}_{\text{wg}}^3$.

Proof.

- a) Since $X \in \mathbf{Cat}_{\text{wg}}^3$, $X_0 \in \mathbf{Cat}_{\text{hd}}^2$ therefore for each $k \geq 0$

$$(N^{(2)}X)_{k0} = X_{0k} \in \mathbf{Cat}_{\text{hd}}.$$

Further, since $p^{(3)}X \in \mathbf{Cat}_{\text{wg}}^2$, for each $k \geq 0$

$$\bar{p}J_2(N^{(2)}X)_k = (p^{(3)}X)_k^{(2)}$$

is the nerve of a category. It follows from Lemma 3.14 applied to $(N^{(2)}X)_k$ that $(N^{(2)}X)_k \in \mathbf{Cat}_{\text{wg}}^2$.

- b) By hypothesis, $X_s \in \mathbf{Cat}^2$ is such that $X_{s0} \in \mathbf{Cat}_{\text{hd}}$ and $\bar{p}J_2X_s$ is the nerve of a category. Thus by Lemma 3.14, $X_s \in \mathbf{Cat}_{\text{wg}}^2$.

Also by hypothesis $p^{(3)}X \in \mathbf{Cat}_{\text{wg}}^2$. To show that $X \in \mathbf{Cat}_{\text{wg}}^3$ it remains to prove that the map

$$\hat{\mu}_s : X_1 \times_{X_0} \cdots \times_{X_0} X_1 \rightarrow X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1$$

is a 2-equivalence. We first show this for $s = 2$, the case $s > 2$ being similar. We first show that it is a local equivalence. By part a) $(N^{(2)}X)_1 \in \mathbf{Cat}_{\text{wg}}^2$. Thus there is an equivalence of categories

$$X_{11} \times_{X_{01}} X_{11} \rightarrow X_{11} \times_{X_{01}^d} X_{11} = X_{11} \times_{(d^{(2)}p^{(2)}X_0)_1} X_{11}. \quad (12)$$

From hypothesis ii) by Remark 3.9 using the fact that $pp^{(2)}X_{s0} = X_{s0}^d$ we have

$$X_{20}^d = (X_{10} \times_{X_{00}} X_{10})^d \cong X_{10}^d \times_{X_{00}^d} X_{10}^d.$$

Let $(a, b), (c, d) \in X_{10}^d \times_{X_{00}^d} X_{10}^d$. By (12) there is an equivalence of categories

$$\begin{aligned} (X_1 \times_{X_0} X_1)((a, b), (c, d)) &= \\ &= X_1(a, c) \times_{X_0(\partial_0 a, \partial_0 c)} X_1(b, d) \rightarrow X_1(a, c) \times_{(d^{(2)} p^{(2)} X_0)(\tilde{\partial}_0 a, \tilde{\partial}_0 c)} X_1(b, d) \end{aligned} \quad (13)$$

On the other hand, since $p^{(2)} X_0 \in \mathbf{Cat}_{\text{hd}}$, $p^{(2)} X_0(\tilde{\partial}_0 a, \tilde{\partial}_0 c)$ is the one-element set. Therefore

$$\begin{aligned} X_1(a, c) \times_{(d^{(2)} p^{(2)} X_0)(\tilde{\partial}_0 a, \tilde{\partial}_0 c)} X_1(b, d) &= \\ &= X_1(a, c) \times X_1(b, d) = (X_1 \times_{X_0^d} X_1)((a, b), (c, d)). \end{aligned} \quad (14)$$

From (13) and (14) we conclude that $\hat{\mu}_2$ is a local equivalence. Further, by hypothesis ii), there is an equivalence of categories

$$\begin{aligned} p^{(2)} \hat{\mu}_2 : p^{(2)}(X_1 \times_{X_0} X_1) &= p^{(2)} X_1 \times_{p^{(2)} X_0} p^{(2)} X_1 \xrightarrow{\sim} \\ &\rightarrow p^{(2)} X_1 \times_{(p^{(2)} X_0)^d} p^{(2)} X_1 = p^{(2)}(X_1 \times_{X_0^d} X_1). \end{aligned}$$

In conclusion, $\hat{\mu}_2$ is a 2-equivalence, as required. \square

Proposition 3.16.

- a) The functor $N^{(2)} : \mathbf{Cat}^n \rightarrow [\Delta^{op}, \mathbf{Cat}^{n-1}]$ restricts to a functor $N^{(2)} : \mathbf{Cat}_{\text{wg}}^n \rightarrow [\Delta^{op}, \mathbf{Cat}_{\text{wg}}^{n-1}]$.
- b) Let $X \in \mathbf{Cat}^n$ be such that
 - i) $X_0 \in \mathbf{Cat}_{\text{hd}}^{n-1}$, $X_{s0} \in \mathbf{Cat}_{\text{hd}}^{n-2}$.
 - ii) $\bar{p}J_n X \in N_{(n-1)} \mathbf{Cat}_{\text{wg}}^{n-1}$ for all $s \geq 0$.

Then $X \in \mathbf{Cat}_{\text{wg}}^n$.

Proof. By induction on n . For $n = 2, 3$ see Lemmas 3.14 and 3.15. Suppose, inductively, that it holds for $(n-1)$.

a) Clearly $X_k^{(2)} \in \mathbf{Cat}^{n-1}$; we show that $X_k^{(2)}$ satisfies the inductive hypothesis b) and thus conclude that $X_k^{(2)} \in \mathbf{Cat}_{\text{wg}}^{n-1}$.

We have $(X_k^{(2)})_0 = X_{0k} \in \mathbf{Cat}_{\text{hd}}^{n-2}$ since $X_0 \in \mathbf{Cat}_{\text{hd}}^{n-1}$ (as $X \in \mathbf{Cat}_{\text{wg}}^n$). Further,

$$(X_k^{(2)})_{s0} = X_{sk0} \in \mathbf{Cat}_{\text{hd}}^{n-3}$$

since $X_{sk} \in \mathbf{Cat}_{\text{wg}}^{n-2}$ (as $X_s \in \mathbf{Cat}_{\text{wg}}^{n-1}$ because $X \in \mathbf{Cat}_{\text{wg}}^n$). Thus condition i) in the inductive hypothesis b) holds for $X_k^{(2)}$. To show that condition ii) holds, notice that

$$\bar{p}J_{n-1} X_k^{(2)} = N_{(n-1)}(p^{(n)} X)_k^{(2)} \quad (15)$$

In fact, for all $(r_1, \dots, r_{n-2}) \in \Delta^{n-2op}$,

$$\begin{aligned} & (\bar{p}J_{n-1}X_k^{(2)})_{r_1 \dots r_{n-2}} = p(X_k^{(2)})_{r_1 \dots r_{n-2}} = \\ & = pX_{r_1, k, r_2 \dots r_{n-2}} = (\bar{p}J_{n-2}X_{r, k})_{r_2 \dots r_{n-2}} = \\ & = (N_{(n-2)}p^{(n-2)}X_{r, k})_{r_2 \dots r_{n-2}} = \\ & = (N_{(n-2)}((p^{(n)}X)_k^{(2)})_{r_1})_{r_2 \dots r_{n-2}} = \\ & = (N_{(n-1)}(p^{(n)}X)_k^{(2)})_{r_1 \dots r_{n-2}} . \end{aligned}$$

Since this holds for all r_1, \dots, r_{n-2} , (15) follows

By induction hypothesis a) applied to $p^{(n)}X$, $(p^{(n)}X)_k^{(2)} \in \mathbf{Cat}_{\mathbf{wg}}^{n-2}$. Therefore (15) means that $X_k^{(2)} \in \mathbf{Cat}^{n-1}$ satisfies condition ii) in the inductive hypothesis b). Thus we conclude that $X_k^{(2)} \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$ proving a).

b) Suppose, inductively, that the statement holds for $n-1$ and let X be as in the hypothesis. For each $s \geq 0$ consider $X_s \in \mathbf{Cat}^{n-1}$. By hypothesis, $X_{s0} \in \mathbf{Cat}_{\mathbf{hd}}^{n-2}$ and

$$\bar{p}J_{n-1}X_s = (\bar{p}J_nX)_s \in N_{(n-2)}\mathbf{Cat}_{\mathbf{wg}}^{n-2}$$

since $\bar{p}J_nX \in N_{(n-1)}\mathbf{Cat}_{\mathbf{wg}}^{n-1}$. Thus X_s satisfies the induction hypothesis and we conclude that $X_s \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$. Further, for each k_1, \dots, k_{n-2} we have

$$\begin{aligned} & (N_{(n-1)}\overline{p^{(n-1)}X})_{k_1 \dots k_{n-2}} = (N_{(n-2)}p^{(n-1)}X_{k_1})_{k_2 \dots k_{n-2}} = \\ & = (\bar{p}J_{n-1}X_{k_1})_{k_2 \dots k_{n-2}} = pX_{k_1 \dots k_{n-2}} = (\bar{p}J_nX)_{k_1 \dots k_{n-2}} . \end{aligned}$$

Since, by hypothesis, $\bar{p}J_nX \in N_{(n-1)}\mathbf{Cat}_{\mathbf{wg}}^{n-1}$, we conclude that $\overline{p^{(n-1)}X} \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$. We can therefore define $p^{(n)}X = \overline{p^{(n-1)}X} \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$.

To prove that $X \in \mathbf{Cat}_{\mathbf{wg}}^n$ it remains to prove that the induced Segal maps

$$\hat{\mu}_s : X_1 \times_{X_0} \cdots^s \times_{X_0} X_1 \rightarrow X_1 \times_{X_0^d} \cdots^s \times_{X_0^d} X_1$$

are $(n-1)$ -equivalences for all $s \geq 2$. We prove this for $s=2$ the case $s > 2$ being similar. We claim that $X_k^{(2)} \in \mathbf{Cat}^{n-1}$ satisfies the inductive hypothesis b). In fact, $(X_k^{(2)})_0 = X_{0k} \in \mathbf{Cat}_{\mathbf{hd}}^{n-2}$ since $X_0 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$; for each $s \geq 0$, $(X_k^{(2)})_{s0} = X_{sk0} \in \mathbf{Cat}_{\mathbf{hd}}^{n-3}$ since, from above, $X_s \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$.

Also, from a) and the fact that, by hypothesis, $\bar{p}J_nX \in N_{(n-1)}\mathbf{Cat}_{\mathbf{wg}}^{n-1}$, we conclude that

$$\bar{p}J_{n-1}X_k^{(2)} \in N_{(n-2)}\mathbf{Cat}_{\mathbf{wg}}^{n-2} .$$

Thus $X_k^{(2)}$ satisfies the inductive hypothesis b) and we conclude that $X_k^{(2)} \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$. It follows that the induced Segal map

$$X_{1k} \times_{X_{0k}} X_{1k} \rightarrow X_{1k} \times_{X_{0k}^d} X_{1k} = X_{1k} \times_{(p^{(2,n-1)}X_0)_k} X_{1k} \quad (16)$$

is a $(n-2)$ -equivalence. Since $p^{(n)}X \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$, using Remark 3.9 and the fact that $(p^{(n)}X)_{s_0}^d = (p^{(n-2)}p^{(n-1)}X_{s_0})^d = X_{s_0}^d$ we obtain

$$(X_{10} \times_{X_{00}} X_{10})^d = X_{10}^d \times_{X_{00}^d} X_{10}^d .$$

Let $(a, b), (c, d) \in (X_{10} \times_{X_{00}} X_{10})^d = X_{10}^d \times_{X_{00}^d} X_{10}^d$. By (16) there is a $(n-2)$ -equivalence

$$\begin{aligned} (X_1 \times_{X_0} X_1)((a, b), (c, d)) &= X_1(a, c) \times_{X_0(\partial_0 a, \partial_0 c)} X_1(b, d) \rightarrow \\ &\rightarrow X_1(a, c) \times_{(p^{(2, n-1)}X_0)(\tilde{\partial}_0 a, \tilde{\partial}_0 c)} X_1(b, d) . \end{aligned} \quad (17)$$

On the other hand, $p^{(2, n-1)}X_0 \in \mathbf{Cat}_{\mathbf{hd}}$ is an equivalence relation, therefore

$$p^{(2, n-1)}X_0(\tilde{\partial}_0 a, \tilde{\partial}_0 c)$$

is the one-element set. It follows that

$$\begin{aligned} X_1(a, c) \times_{(p^{(2, n-1)}X_0)(\tilde{\partial}_0 a, \tilde{\partial}_0 c)} X_1(b, d) &\cong \\ &\cong X_1(a, c) \times X_1(b, d) \cong (X_1 \times_{X_0^d} X_1)((a, b), (c, d)) . \end{aligned} \quad (18)$$

Thus (17) and (18) imply that $\hat{\mu}_2$ is a local $(n-2)$ -equivalence.

To show that $\hat{\mu}_2$ is a $(n-1)$ -equivalence it remains to prove that $p^{(n-1)}\hat{\mu}_2$ is a $(n-2)$ -equivalence. Since from above, $p^{(n)}X = \overline{p^{(n-1)}X} \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$, we have

$$\begin{aligned} p^{(n-1)}\hat{\mu}_2 : p^{(n-1)}(X_1 \times_{X_0} X_1) &\cong p^{(n-1)}X_1 \times_{p^{(n-1)}X_0} p^{(n-1)}X_1 \rightarrow \\ &\rightarrow p^{(n-1)}X_1 \times_{(p^{(n-1)}X_0)^d} p^{(n-1)}X_1 = p^{(n-1)}(X_1 \times_{X_0^d} X_1) . \end{aligned}$$

is a $(n-2)$ -equivalence, as required. \square

4. SEGALIC PSEUDO-FUNCTORS AND THEIR STRICTIFICATION

In this section we introduce the category $\mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}]$ of Segalic pseudo-functors and we prove in Theorem 4.5 that the strictification of a Segalic pseudo-functor is a weakly globular n -fold category.

After giving the definition of Segalic pseudo-functors in 4.1, we show in Proposition 4.3 that if an n -fold category, viewed as a diagram in $[\Delta^{n-1^{op}}, \mathbf{Cat}]$ is levelwise equivalent to a Segalic pseudo-functor, then it is weakly globular. We show there that the strictification machinery, when applied to a Segalic pseudo-functor, produces an n -fold category satisfying the hypotheses of Proposition 4.3 (when viewed as a diagram in $[\Delta^{n-1^{op}}, \mathbf{Cat}]$). We conclude that its strictification is a weakly globular n -fold category.

4.1. The idea of Segalic pseudo-functors. The category of Segalic pseudo-functors is a full subcategory of the category $\mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}]$ of pseudo-functors and pseudo-natural transformations [10].

A topological intuition about an object of $\mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}]$ is that it consists of categories $X_{\underline{k}}$ for each object \underline{k} of $\Delta^{n-1^{op}}$ together with multi-simplicial face and degeneracy maps satisfying the multi-simplicial identities not as equalities but as isomorphisms, and these isomorphisms satisfy coherence axioms. Guided by this intuition, we generalize to certain pseudo-functors the multi-simplicial notion of Segal map.

For this purpose, consider a functor $H \in [\Delta^{n-1^{op}}, \mathbf{Cat}]$. For each $\underline{k} = (k_1, \dots, k_{n-1}) \in \Delta^{n-1^{op}}$ and $1 \leq i \leq n-1$ we have

$$H(k_1, \dots, k_{i-1}, -, k_{i+1}, \dots, k_{n-1}) \in [\Delta^{op}, \mathbf{Cat}]$$

and there is a corresponding Segal map for each $k_i \geq 2$

$$H_{\underline{k}} \rightarrow H_{\underline{k}(1,i)} \times_{H_{\underline{k}(0,i)}} \cdots \times_{H_{\underline{k}(0,i)}} H_{\underline{k}(1,i)} \quad (19)$$

identified by the commuting diagram

$$\begin{array}{c} H_{\underline{k}} \\ \begin{array}{ccc} \swarrow \nu_1 & \downarrow \nu_2 & \searrow \nu_k \\ H_{\underline{k}(1,i)} & H_{\underline{k}(1,i)} & \cdots & H_{\underline{k}(1,i)} \\ \begin{array}{ccc} \swarrow d_1 & \searrow d_0 & \downarrow d_1 & \searrow d_0 \\ H_{\underline{k}(0,i)} & H_{\underline{k}(0,i)} & \cdots & H_{\underline{k}(0,i)} \end{array} \end{array} \end{array} \quad (20)$$

If H is not a functor but a pseudo-functor $H \in \mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}]$, diagram (20) no longer commutes but pseudo-commutes and thus we can no longer define Segal maps. However, if $H_{\underline{k}(0,i)}$ is a discrete category, then diagram (20) commutes and therefore we can define Segal maps for H .

In the definition of Segalic pseudo-functor we require the above discreteness conditions to be satisfied so as to be able to speak about Segal maps and then we require all Segal maps to be isomorphisms.

The last condition in the definition of Segalic pseudo-functor is about the existence of a truncation functor. Applying the isomorphism classes of objects functor $p : \mathbf{Cat} \rightarrow \mathbf{Set}$ to any pseudo-functor in $\mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}]$ produces a strict functor; that is, there is a functor

$$\bar{p} : \mathbf{Ps}[\Delta^{n^{op}}, \mathbf{Cat}] \rightarrow [\Delta^{n-1^{op}}, \mathbf{Set}]$$

such that, for all $\underline{k} \in \Delta^{n-1^{op}}$

$$(\bar{p}X)_{\underline{k}} = pX_{\underline{k}}.$$

For X to be a Segalic pseudo-functor we require $\bar{p}X$ to be a weakly globular n -fold category (more precisely, in the image of $J_n : \mathbf{Cat}_{\mathbf{wg}}^n \rightarrow [\Delta^{n-1^{op}}, \mathbf{Cat}]$). Thus we have a functor

$$p^{(n+1)} : \mathbf{SegPs}[\Delta^{n^{op}}, \mathbf{Cat}] \rightarrow \mathbf{Cat}_{\mathbf{wg}}^n.$$

4.2. Segal maps for special pseudo-functors. Let $H \in \text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ be such that $H_{\underline{k}(0,i)}$ is discrete for all $\underline{k} \in \Delta^{n-1^{op}}$ and all $i \geq 0$. Then the following diagram commutes, for each $k_i \geq 2$.

$$\begin{array}{c}
 H_{\underline{k}} \\
 \swarrow \nu_1 \quad \searrow \nu_2 \quad \cdots \quad \searrow \nu_k \\
 H_{\underline{k}(1,i)} \quad H_{\underline{k}(1,i)} \quad \cdots \quad H_{\underline{k}(1,i)} \\
 \swarrow d_1 \quad \searrow d_0 \quad \swarrow d_1 \quad \searrow d_0 \quad \cdots \quad \swarrow d_1 \quad \searrow d_0 \\
 H_{\underline{k}(0,i)} \quad H_{\underline{k}(0,i)} \quad H_{\underline{k}(0,i)} \cdots H_{\underline{k}(0,i)} \quad H_{\underline{k}(0,i)}
 \end{array}$$

There is therefore a unique Segal map

$$H_{\underline{k}} \rightarrow H_{\underline{k}(1,i)} \times_{H_{\underline{k}(0,i)}} \cdots \times_{H_{\underline{k}(0,i)}}^{k_i} H_{\underline{k}(1,i)} .$$

4.3. The definition of Segalic pseudo-functor.

Definition 4.1. We define the subcategory $\text{SegPs}[\Delta^{n^{op}}, \text{Cat}]$ of $\text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ as follows:

For $n = 1$, $H \in \text{SegPs}[\Delta^{op}, \text{Cat}]$ if H_0 is discrete and the Segal maps are isomorphisms: that is, for all $k \geq 2$

$$H_k \cong H_1 \times_{H_0} \cdots \times_{H_0}^k H_1$$

Note that, since p commutes with pullbacks over discrete objects, there is a functor

$$\begin{aligned}
 p^{(2)} : \text{SegPs}[\Delta^{op}, \text{Cat}] &\rightarrow \text{Cat} , \\
 (p^{(2)} X)_k &= pX_k .
 \end{aligned}$$

That is the following diagram commutes:

$$\begin{array}{ccc}
 \text{SegPs}[\Delta^{op}, \text{Cat}] & \xrightarrow{\quad} & \text{Ps}[\Delta^{op}, \text{Cat}] \\
 p^{(2)} \downarrow & & \downarrow \bar{p} \\
 \text{Cat} & \xrightarrow{\quad} & [\Delta^{op}, \text{Set}]
 \end{array}$$

When $n > 1$, $\text{SegPs}[\Delta^{n^{op}}, \text{Cat}]$ is the full subcategory of $\text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ whose objects H satisfy the following:

a) *Discreteness condition:* $H_{\underline{k}(0,i)}$ is discrete for all $\underline{k} \in \Delta^{n-1^{op}}$ and $1 \leq i \leq n$.

b) *Segal condition:* All Segal maps are isomorphisms

$$H_{\underline{k}} \cong H_{\underline{k}(1,i)} \times_{H_{\underline{k}(0,i)}} \cdots \times_{H_{\underline{k}(0,i)}}^{k_i} H_{\underline{k}(1,i)}$$

for all $\underline{k} \in \Delta^{n-1^{op}}$, $1 \leq i \leq n$ and $k_i \geq 2$.

c) *Truncation functor:* There is a functor

$$p^{(n+1)} : \text{SegPs}[\Delta^{n^{op}}, \text{Cat}] \rightarrow \text{Cat}_{\text{wg}}^n$$

making the following diagram commute:

$$\begin{array}{ccc} \text{SegPs}[\Delta^{n^{op}}, \text{Cat}] & \hookrightarrow & \text{Ps}[\Delta^{n^{op}}, \text{Cat}] \\ p^{(n+1)} \downarrow & & \downarrow \bar{p} \\ \text{Cat}_{\text{wg}}^n & \xrightarrow{N(n)} & [\Delta^{n^{op}}, \text{Set}] \end{array}$$

Lemma 4.2. *Let $X \in \text{SegPs}[\Delta^{n^{op}}, \text{Cat}]$ $n \geq 2$. Then for each $j \geq 0$ $X_{j*} \in \text{SegPs}[\Delta^{n-1^{op}}, \text{Cat}]$.*

Proof. By induction on n . Let $X \in \text{SegPs}[\Delta^{2^{op}}, \text{Cat}]$. Since $X \in \text{Ps}[\Delta^{2^{op}}, \text{Cat}]$, for each $j \geq 0$ $X_{j*} \in \text{Ps}[\Delta^{op}, \text{Cat}]$. By definition of Segalic pseudo-functor, X_{j0} is discrete and for each $r \geq 2$

$$X_{jr} \cong X_{j1} \times_{X_{j0}} \cdots \times_{X_{j0}} X_{j1}.$$

By definition this means that $X_{j*} \in \text{SegPs}[\Delta^{op}, \text{Cat}]$. Suppose, inductively, that the lemma holds for $(n-1)$ and let $X \in \text{SegPs}[\Delta^{n^{op}}, \text{Cat}]$. For each $j \geq 0$, $X_{j*} \in \text{Ps}[\Delta^{n-1^{op}}, \text{Cat}]$.

Given $\underline{r} \in \Delta^{n-1^{op}}$ denote $\underline{k} = (j, \underline{r}) \in \Delta^{n^{op}}$. Then for each $1 \leq i \leq n-2$,

$$X_{\underline{k}(i+1,0)} = (X_j)_{\underline{r}(i,0)}$$

is discrete since $X \in \text{SegPs}[\Delta^{n^{op}}, \text{Cat}]$; further, by hypothesis there are isomorphisms:

$$\begin{aligned} (X_j)_{\underline{r}} &= X_{\underline{k}} \cong X_{\underline{k}(i+1,1)} \times_{X_{\underline{k}(i+1,0)}} \cdots \times_{X_{\underline{k}(i+1,0)}} X_{\underline{k}(i+1,1)} \cong \\ &\cong (X_j)_{\underline{r}(i,1)} \times_{(X_j)_{\underline{r}(i,0)}} \cdots \times_{(X_j)_{\underline{r}(i,0)}} (X_j)_{\underline{r}(i,1)}. \end{aligned}$$

To show that $X_j \in \text{SegPs}[\Delta^{n-1^{op}}, \text{Cat}]$ it remains to show that $p^{(n)}X_j \in \text{Cat}_{\text{wg}}^{n-1}$ where

$$(p^{(n)}X_j)_{\underline{r}} = pX_{j\underline{r}}$$

for each $\underline{r} \in \Delta^{n-1^{op}}$.

Since $X \in \text{SegPs}[\Delta^{n^{op}}, \text{Cat}]$, by definition $p^{(n+1)}X \in \text{Cat}_{\text{wg}}^n$ where $(p^{(n+1)}X)_{\underline{k}} = pX_{\underline{k}}$ for all $\underline{k} \in \Delta^{n^{op}}$. We also observe that, for each $j \geq 0$

$$(p^{(n+1)}X)_j = p^{(n)}X_j \tag{21}$$

since, for each $\underline{r} \in \Delta^{n-1^{op}}$,

$$(p^{(n+1)}X)_{j\underline{r}} = pX_{j\underline{r}} = (p^{(n)}X_j)_{\underline{r}}.$$

Since $p^{(n+1)}X \in \text{Cat}_{\text{wg}}^n$, $(p^{(n+1)}X)_j \in \text{Cat}_{\text{wg}}^{n-1}$ so by (21) we conclude that $p^{(n)}X_j \in \text{Cat}_{\text{wg}}^{n-1}$ as required. \square

4.4. Segalic pseudo-functors and n -fold categories. In the following proposition we show that if an n -fold category is levelwise equivalent (as a diagram in $[\Delta^{n-1^{op}}, \mathbf{Cat}]$) to a Segalic pseudo-functor, then it is a weakly globular n -fold category.

Proposition 4.3. *Let $H \in \mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}]$ and let $L \in \mathbf{Cat}^n$ be such that there is an equivalence of categories $(J_n L)_{\underline{k}} \simeq H_{\underline{k}}$ for all $\underline{k} \in \Delta^{n-1^{op}}$, then*

- a) $L \in \mathbf{Cat}_{\mathbf{wg}}^n$.
- b) *If, further, $H_{\underline{k}} \in \mathbf{Cat}_{\mathbf{hd}}$ for all \underline{k} and $p^{(n)}H \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$, then $L \in \mathbf{Cat}_{\mathbf{hd}}^n$.*

Proof. By induction on n . For $n = 2$, if $H \in \mathbf{SegPs}[\Delta^{op}, \mathbf{Cat}]$, then by definition H_0 is discrete; thus $L_0 \in \mathbf{Cat}_{\mathbf{hd}}$. Also, since $L_k \simeq H_k$ for all $k \in \Delta^{op}$, $pL_k \cong pH_k$, and therefore $\bar{p}L \cong \bar{p}H = p^{(2)}H$ is the nerve of a category. So by Proposition 3.16 b) $L \in \mathbf{Cat}_{\mathbf{wg}}^2$.

If, further, $H_k \in \mathbf{Cat}_{\mathbf{hd}}$ for all k and $p^{(2)}H \in \mathbf{Cat}_{\mathbf{hd}}$, then $L_1 \in \mathbf{Cat}_{\mathbf{hd}}$ (since $L_1 \sim H_1$) and $p^{(2)}L = p^{(2)}H \in \mathbf{Cat}_{\mathbf{hd}}$. Therefore, by Corollary 3.11, $L \in \mathbf{Cat}_{\mathbf{hd}}^2$.

Suppose, inductively, that the lemma holds for $n - 1$ and let L and H be as in the hypothesis a).

We are going to show that $L \in \mathbf{Cat}^n$ satisfies the hypotheses of Proposition 3.16 b) which then implies that $L \in \mathbf{Cat}_{\mathbf{wg}}^n$.

Let $\underline{r} \in \Delta^{n-2^{op}}$ and denote $\underline{k} = (i, \underline{r}) \in \Delta^{n-1^{op}}$. By hypothesis, there are equivalences of categories

$$(J_{n-1}L_i)_{\underline{r}} = J_n L_{\underline{k}} \cong H_{\underline{k}} = (H_i)_{\underline{r}}.$$

Since $L \in \mathbf{Cat}^n$, $L_i \in \mathbf{Cat}^{n-1}$ and since $H \in \mathbf{SegPs}[\Delta^{n^{op}}, \mathbf{Cat}]$, by Lemma 4.2, $H_i \in \mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}]$.

Thus $H_{i\bullet}$ and $L_{i\bullet}$ satisfy the inductive hypothesis a) and we conclude that $L_{i\bullet} \in \mathbf{Cat}_{\mathbf{wg}}^{n-1}$. In particular, this implies that $L_{i0} \in \mathbf{Cat}_{\mathbf{hd}}^{n-2}$. Thus, by Proposition 3.16 b), to show that $L \in \mathbf{Cat}_{\mathbf{wg}}^n$ it is enough to prove that $L_{0\bullet} \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ and that $\bar{p}J_n L \in N_{(n-1)}\mathbf{Cat}_{\mathbf{wg}}^{n-1}$. We have $(H_{0\bullet})_{\underline{k}} = H_{\underline{k}(0,0)}$ discrete and

$$p^{(n-1)}H_{0\bullet} = (p^{(n)}H)_0 \in \mathbf{Cat}_{\mathbf{hd}}^{n-2}$$

since by hypothesis $p^{(n)}H \in \mathbf{Cat}_{\mathbf{wg}}^n$ as $H \in \mathbf{SegPs}[\Delta^{n^{op}}, \mathbf{Cat}]$. Thus $H_{0\bullet}$ and $L_{0\bullet}$ satisfy the inductive hypothesis b) and we conclude that $L_{0\bullet} \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$. For each $\underline{k} \in \Delta^{n-2^{op}}$,

$$(\bar{p}J_n L)_{\underline{k}} = pL_{\underline{k}} \cong pH_{\underline{k}} = (\bar{p}J_n H)_{\underline{k}}.$$

Since $p^{(n)}H \in \mathbf{Cat}_{\mathbf{wg}}^n$, this means that $\bar{p}J_n L \in N_{(n-1)}\mathbf{Cat}_{\mathbf{wg}}^{n-1}$. By Proposition 3.16 b), we conclude that $L \in \mathbf{Cat}_{\mathbf{wg}}^n$, proving a) at step n .

Suppose that H is as in b). By Corollary 3.11, to show that $L \in \mathbf{Cat}_{\mathbf{hd}}^n$, it is enough to show that $L_1 \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$ and $p^{(n)}L \in \mathbf{Cat}_{\mathbf{hd}}^{n-1}$. For

all $\underline{k} \in \Delta^{n-2^{op}}$ there is an equivalence of categories

$$(L_{1\bullet})_{\underline{k}} \simeq (H_{1\bullet})_{\underline{k}}$$

therefore, since by hypothesis, $(H_{1\bullet})_{\underline{k}} \in \mathbf{Cat}_{\text{hd}}$, also $(L_{1\bullet})_{\underline{k}} \in \mathbf{Cat}_{\text{hd}}$. Further, since $p^{(n)}H \in \mathbf{Cat}_{\text{hd}}^{n-1}$, then

$$p^{(n-1)}H_{1\bullet} = (p^{(n)}H)_{1\bullet} \in \mathbf{Cat}_{\text{hd}}^{n-2}.$$

Thus $L_{1\bullet}$ and $H_{1\bullet}$ satisfy induction hypothesis and we conclude that $L_{1\bullet} \in \mathbf{Cat}_{\text{hd}}^{n-1}$. Finally,

$$p^{(n)}L = p^{(n)}H \in \mathbf{Cat}_{\text{hd}}^{n-1}.$$

Thus by Corollary 3.11 we conclude that $L \in \mathbf{Cat}_{\text{hd}}^n$. \square

4.5. Strictification of Segalic pseudo-functors. In this section we prove our main result, Theorem 4.5, that the strictification functor applied to the category of Segalic pseudo-functors gives a weakly globular n -fold category. The strategy to prove this result is to show that the strictification of a Segalic pseudo-functor is an n -fold category and that it satisfies the hypotheses of Proposition 4.3.

Lemma 4.4. *Let T be the monad corresponding to the adjunction given by the forgetful functor*

$$U : [\Delta^{n^{op}}, \mathbf{Cat}] \rightarrow [\text{ob}(\Delta^{n^{op}}), \mathbf{Cat}]$$

and its left adjoint. Let $H \in \text{SegPs}[\Delta^{n^{op}}, \mathbf{Cat}]$, then

- a) *The pseudo T -algebra corresponding to H has structure map $h : TUH \rightarrow H$ as follows:*

$$(TUH)_{\underline{k}} = \coprod_{\underline{r} \in \Delta^n} \Delta^n(\underline{k}, \underline{r}) \times H_{\underline{r}} = \coprod_{\underline{r} \in \Delta^n} \coprod_{\Delta^n(\underline{k}, \underline{r})} H_{\underline{r}}.$$

If $f \in \Delta^n(\underline{k}, \underline{r})$, let

$$i_{\underline{r}} = \coprod_{\Delta^n(\underline{k}, \underline{r})} H_{\underline{r}} \rightarrow \coprod_{\underline{r} \in \Delta^n} \coprod_{\Delta^n(\underline{k}, \underline{r})} H_{\underline{r}} = (TUH)_{\underline{r}}$$

$$j_f : H_{\underline{r}} \rightarrow \coprod_{\Delta^n(\underline{k}, \underline{r})} H_{\underline{r}},$$

then

$$h_{\underline{k}} i_{\underline{r}} j_f = H(f).$$

- b) *There are functors for each $\underline{k} \in \Delta^n$, $0 \leq i \leq n$,*

$$\partial_{i1}, \partial_{i0} : (TUH)_{\underline{k}(1,i)} \rightrightarrows (TUH)_{\underline{k}(0,i)}$$

such that the following diagram commutes

$$\begin{array}{ccc}
 (TUH)_{\underline{k}(1,i)} & \xrightarrow{h_{\underline{k}(1,i)}} & H_{\underline{k}(1,i)} \\
 \partial_{i0} \downarrow \quad \partial_{i1} \downarrow & & d_{i0} \downarrow \quad d_{i1} \downarrow \\
 (TUH)_{\underline{k}(0,i)} & \xrightarrow{h_{\underline{k}(0,i)}} & H_{\underline{k}(0,i)}
 \end{array} \tag{22}$$

c) For all $\underline{k} = (k_1, \dots, k_n) \in \Delta^n$ there are isomorphisms

$$(TUH)_{\underline{k}} = (TUH)_{\underline{k}(1,i)} \times_{(TUH)_{\underline{k}(0,i)}} \cdots \times_{(TUH)_{\underline{k}(0,i)}}^{k_i} (TUH)_{\underline{k}(1,i)} .$$

d) The morphism $h_{\underline{k}} : (TUH)_{\underline{k}} \rightarrow H_{\underline{k}}$ is given by

$$h_{\underline{k}} = (h_{\underline{k}(1,i)}, \dots, h_{\underline{k}(1,i)})$$

Proof.

a) From the general correspondence between pseudo T -algebras and pseudo-functors, the pseudo T -algebra corresponding to H has a structure map $h : TUH \rightarrow H$ as stated. The rest follows from the fact that, if X is a set and \mathcal{C} is a category, $X \times \mathcal{C} \cong \coprod_X \mathcal{C}$.

b) Let $\nu_j : [0] \rightarrow [1]$, $\nu_j(0) = 0$. $\nu_j(1) = i$ for $j = 0, 1$ and let $\delta_{ij} : \underline{k}(0, i) \rightarrow \underline{k}(1, i)$ be given by

$$\delta_{ij}(k_s) = \begin{cases} k_s, & s \neq i; \\ \nu_j(k_i), & s = i. \end{cases}$$

Given $f \in \Delta^n(\underline{k}(1, i), \underline{r})$ let j_f and $i_{\underline{r}}$ be the corresponding coproduct injections as in a). Let

$$\partial_{ij} : (TUH)_{\underline{k}(1,i)} \rightarrow (TUH)_{\underline{k}(0,i)}$$

be the functors determined by

$$\partial_{ij} i_{\underline{r}} j_f = i_{\underline{r}} j_f \delta_{ij} .$$

From a), we have

$$\begin{aligned}
 h_{\underline{k}(0,i)} \partial_{ij} i_{\underline{r}} j_f &= h_{\underline{k}(0,i)} i_{\underline{r}} j_f \delta_{ij} = H(f \delta_{ij}) \\
 d_{ij} h_{\underline{k}(1,i)} i_{\underline{r}} j_f &= H(\delta_{ij}) H(f) .
 \end{aligned}$$

Since $H \in \text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ and $H_{\underline{k}(0,i)}$ is discrete, it is $H(f \delta_{ij}) = H(\delta_{ij}) H(f)$ so that, from above,

$$h_{\underline{k}(0,i)} \partial_{ij} i_{\underline{r}} j_f = d_{ij} h_{\underline{k}(1,i)} i_{\underline{r}} j_f$$

for each \underline{r}, f . We conclude that

$$h_{\underline{k}(0,i)} \partial_{ij} = d_{ij} h_{\underline{k}(1,i)} .$$

That is, diagram (22) commutes.

c) Since, for each $k_i \geq 2$

$$[k_i] = [1] \coprod_{[0]}^{k_i} \coprod_{[0]} [1]$$

we have, for each $\underline{k} = (k_1, \dots, k_n) \in \Delta^n$ with $k_i \geq 2$ for all $1 \leq i \leq n$

$$\underline{k} = \underline{k}(1, i) \coprod_{\underline{k}(0, i)}^{k_i} \coprod_{\underline{k}(0, i)} \underline{k}(1, i) .$$

Therefore we have a bijection

$$\Delta^n(\underline{k}, \underline{r}) = \Delta^n(\underline{k}(1, i), \underline{r}) \times_{\Delta^n(\underline{k}(0, i), \underline{r})} \cdots \times_{\Delta^n(\underline{k}(0, i), \underline{r})} \Delta^n(\underline{k}(1, i), \underline{r}) .$$

From the proof of b), the functors $\partial_{ij} : (TUH)_{\underline{k}(1, i)} \rightarrow (TUH)_{\underline{k}(0, i)}$ for $j = 0, 1$ are determined by the functors

$$(\overline{\delta_{ij}}, id) : \Delta^n(\underline{k}(1, i), \underline{r}) \times H_{\underline{r}} \rightarrow \Delta^n(\underline{k}(0, i), \underline{r}) \times H_{\underline{r}}$$

where $\overline{\delta_{ij}}(g) = g\delta_{ij}$ for $g \in \Delta^n(\underline{k}(1, i), \underline{r})$ and

$$\begin{aligned} (TUH)_{\underline{k}(1, i)} &= \coprod_{\underline{r}} \Delta^n(\underline{k}(1, i), \underline{r}) \times H_{\underline{r}} \\ (TUH)_{\underline{k}(0, i)} &= \coprod_{\underline{r}} \Delta^n(\underline{k}(0, i), \underline{r}) \times H_{\underline{r}} . \end{aligned}$$

It follows that

$$\begin{aligned} (TUH)_{\underline{k}(1, i)} \times_{(TUH)_{\underline{k}(0, i)}} \cdots \times_{(TUH)_{\underline{k}(0, i)}} (TUH)_{\underline{k}(1, i)} &= \\ \coprod_{\underline{r}} \{ \Delta^n(\underline{k}(1, i), \underline{r}) \times_{\Delta^n(\underline{k}(0, i), \underline{r})} \cdots \times_{\Delta^n(\underline{k}(0, i), \underline{r})} \Delta^n(\underline{k}(1, i), \underline{r}) \} \times H_{\underline{r}} &= \\ \coprod_{\underline{r}} \Delta^n(\underline{k}, \underline{r}) \times H_{\underline{r}} &= (TUH)_{\underline{k}} . \end{aligned}$$

This proves c).

d) From a), $h_k i_r j_f = H(f)$ for $f \in \Delta^n(\underline{k}, \underline{r})$. Let f correspond to $(\delta_1, \dots, \delta_{k_i})$ in the isomorphism

$$\Delta^n(\underline{k}, \underline{r}) = \Delta^n(\underline{k}(1, i), \underline{r}) \times_{\Delta^n(\underline{k}(0, i), \underline{r})} \cdots \times_{\Delta^n(\underline{k}(0, i), \underline{r})} \Delta^n(\underline{k}(1, i), \underline{r}) .$$

Then $j_f = (j_{\delta_1}, \dots, j_{\delta_{k_i}})$. Since

$$H_{\underline{k}} \cong H_{\underline{k}(1, i)} \times_{H_{\underline{k}(0, i)}} \cdots \times_{H_{\underline{k}(0, i)}} H_{\underline{k}(1, i)}$$

then $H(f)$ corresponds to $(H(\delta_1), \dots, H(\delta_{k_i}))$ with $p_i H(f) = H(\delta_i)$. Then for all f we have

$$\begin{aligned} h_{\underline{k}} i_{\underline{r}} j_f &= (H(\delta_1), \dots, H(\delta_{k_i})) = (h_{\underline{k}(1, i)} i_{\underline{r}} j_{\delta_1}, \dots, h_{\underline{k}(1, i)} i_{\underline{r}} j_{\delta_{k_i}}) = \\ &= (h_{\underline{k}(1, i)}, \dots, h_{\underline{k}(1, i)}) i_{\underline{r}} (j_{\delta_1}, \dots, j_{\delta_{k_i}}) = (h_{\underline{k}(1, i)}, \dots, h_{\underline{k}(1, i)}) i_{\underline{r}} j_f . \end{aligned}$$

It follows that $h_{\underline{k}} = (h_{\underline{k}(1, i)}, \dots, h_{\underline{k}(1, i)})$. □

Theorem 4.5. *The strictification functor*

$$St : \mathbf{Ps}[\Delta^{n-1^{op}}, \mathbf{Cat}] \rightarrow [\Delta^{n-1^{op}}, \mathbf{Cat}]$$

restricts to a functor

$$L_n : \mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}] \rightarrow J_n \mathbf{Cat}_{\mathbf{wg}}^n$$

where $N_{(n)} \mathbf{Cat}_{\mathbf{wg}}^n$ denotes the image of the multinerve functor $J_n : \mathbf{Cat}_{\mathbf{wg}}^n \rightarrow [\Delta^{n-1^{op}}, \mathbf{Cat}]$. Further, for each $H \in \mathbf{SegPs}[\Delta^{n-1^{op}}, \mathbf{Cat}]$ and $\underline{k} \in \Delta^{n-1^{op}}$, the map $(L_n H)_{\underline{k}} \rightarrow H_{\underline{k}}$ is an equivalence of categories.

Proof. Let $h : T U H \rightarrow U H$ be as in Section 2.5. As recalled there, to construct the strictification $L = St H$ of a pseudo-functor H we need to factorize $h = g v$ in such a way that for each $\underline{k} \in \Delta^{n-1^{op}}$, $h_{\underline{k}}$ factorizes as

$$(T U H)_{\underline{k}} \xrightarrow{v_{\underline{k}}} L_{\underline{k}} \xrightarrow{g_{\underline{k}}} H_{\underline{k}}$$

with $v_{\underline{k}}$ bijective on objects and $g_{\underline{k}}$ fully faithful. As explained in [28], $g_{\underline{k}}$ is in fact an equivalence of categories.

Since the bijective on objects and fully faithful functors form a factorization system in \mathbf{Cat} , the commutativity of (22) implies that there are functors

$$\tilde{d}_{ij} : L_{\underline{k}(1,i)} \rightrightarrows L_{\underline{k}(0,i)} \quad j = 0, 1$$

such that the following diagram commutes:

$$\begin{array}{ccccc} (T U H)_{\underline{k}(1,i)} & \xrightarrow{v_{\underline{k}(1,i)}} & L_{\underline{k}(1,i)} & \xrightarrow{g_{\underline{k}(1,i)}} & H_{\underline{k}(1,i)} \\ \partial_{i0} \downarrow & & \tilde{d}_{i0} \downarrow & & d_{i0} \downarrow \\ (T U H)_{\underline{k}(0,i)} & \xrightarrow{v_{\underline{k}(0,i)}} & L_{\underline{k}(0,i)} & \xrightarrow{g_{\underline{k}(0,i)}} & H_{\underline{k}(0,i)} \end{array} \quad \begin{array}{c} \partial_{i1} \downarrow \\ \tilde{d}_{i1} \downarrow \\ d_{i1} \downarrow \end{array}$$

By Proposition 4.4, $h_{\underline{k}}$ factorizes as

$$\begin{aligned} (T U H)_{\underline{k}} &= (T U H)_{\underline{k}(1,i)} \times_{(T U H)_{\underline{k}(0,i)}} \cdots \times_{(T U H)_{\underline{k}(0,i)}} (T U H)_{\underline{k}(1,i)} \rightarrow \\ &\xrightarrow{(v_{\underline{k}(1,i)}, \dots, v_{\underline{k}(1,i)})} L_{\underline{k}(1,i)} \times_{L_{\underline{k}(0,i)}} \cdots \times_{L_{\underline{k}(0,i)}} L_{\underline{k}(1,i)} \rightarrow \\ &\xrightarrow{(g_{\underline{k}(1,i)}, \dots, g_{\underline{k}(1,i)})} H_{\underline{k}(1,i)} \times_{H_{\underline{k}(0,i)}} \cdots \times_{H_{\underline{k}(0,i)}} H_{\underline{k}(1,i)} \cong H_{\underline{k}}. \end{aligned}$$

Since $v_{\underline{k}(1,i)}$ and $v_{\underline{k}(0,i)}$ are bijective on objects, so is $(v_{\underline{k}(1,i)}, \dots, v_{\underline{k}(1,i)})$. Since $g_{\underline{k}(1,i)}$, $g_{\underline{k}(0,i)}$ are fully faithful, so is $(g_{\underline{k}(1,i)}, \dots, g_{\underline{k}(1,i)})$. Therefore the above is the factorization of $h_{\underline{k}}$ and we conclude that

$$L_{\underline{k}} \cong L_{\underline{k}(1,i)} \times_{L_{\underline{k}(0,i)}} \cdots \times_{L_{\underline{k}(0,i)}} L_{\underline{k}(1,i)}.$$

Since $L \in [\Delta^{n-1^{op}}, \mathbf{Cat}]$ by Lemma 2.11 this implies that $L \in \mathbf{Cat}^n$. By [28], $L_{\underline{k}} \simeq H_{\underline{k}}$ for all $\underline{k} \in \Delta^{n-1^{op}}$. Therefore, by Lemma 4.3, $L \in \mathbf{Cat}_{\mathbf{wg}}^n$. \square

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